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Locality Game: Network Creation in P2P Systems

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Abstract

We present a game modelling the decentralized creation of networks by selfish node agents, e.g. P2P systems. In [4], Fabrikant et al. introduced such a game where each node pays for the links it builds and the length of the paths to the remaining nodes. Our game provides a different model and takes the positions of the nodes into account. Moreover this game offers a model for the profitable formation of links in a peer-to-peer system. We examine the set of stable solutions - the Nash equilibria - of this game regarding undirected and directed links and compare them to a centrally enforced optimum.

1 Introduction

Network design is an important problem in computer science and operations research. This line of research typically assumes a central authority that constructs the network and has various optimization criteria to fulfill. In practice, however, many networks are actually formed by selfish players who are motivated by their own interests and their own objective function. For instance a peer-to-peer (P2P) system is formed by many players and not by a single authority. Every player pursues the aim of being able to route to all other players efficiently and yet to devote a small amount of memory to storing information about them. This motivates the research of network creation by multiple selfish players. In network settings without coordination, each agent seeks to maximize its individual objective function ignoring the effects of their actions to the overall performance of the network. For this reason, the resulting networks, can be much worse than networks designed by a central entity. To describe the consequences of this lack of coordination, Koutsoupias and Papadimitriou introduced in [5] the term *price of anarchy*, to refer to the ratio of the social cost of the worst Nash equilibrium to the social optimum. A different approach is taken by Elliot Anshelevich et al. in [1], where they study the quality of the best Nash equilibrium and denote the ratio of its cost to the social optimum the *price of stability*.

In this thesis we will study both the price of anarchy and the price of stability with respect to our game and give proofs for their upper and lower bounds. In the game-theoretic model of network creation we propose, the players are nodes, and their strategy choices create a graph. Each node chooses a (possibly empty) subset of the other nodes, and establishes edges to them. The union of these sets of edges is the resulting network graph. The cost to a node of such a strategy selection consists of two parts: the sum of the cost of the edges laid down by this node (the number of edges times a constant $\alpha > 0$, the only parameter in this model), plus the sum of the stretches, i.e. the shortest distance in G divided by the Euclidean distance from the node to all others.

A scenario where no node has the incentive to deviate from its strategy is called a Nash equilibrium and forms a stable solution of the game.

This game models scenarios in which peers wish to communicate and transfer data. Every player wants to establish as few direct links to other players as possible and yet have a short delay.

This thesis is organized as follows: We begin with a survey of related work in Section 2. Section 3 introduces the model that will be used throughout this document. Section 4 contains our results concerning the undirected case of our game, whereas the directed case is analysed in Section 5 along with a proof for the upper bound on the price of anarchy. We conclude our work in Section 6, where we also give some directions for future research projects.

2 Related Work

Network design games have been studied in a wide range of models. Fabrikant et al. suggest in [4] a game in the context of communication networks where nodes pay for the links they establish and benefit from short paths. Their main focus was on analyzing the price of anarchy of the game, presenting proofs for its upper and lower bound as well as the tree conjecture stating that there exists a constant A, such that for any $\alpha > A$, all non transient Nash Equilibria are trees.

In [6], Albers et al. disprove the tree conjecture and they improve the lower and upper bound of the price of anarchy. Additionally they study variations of the game, which form an extension of Fabrikant et al.'s game and enable modelling of different traffic load between players and allow players to buy only a fraction of an edge.

In contrast to Fabrikant, where links are generated unilaterally and link cost are carried by only one of its endpoints, Corbo and Parkes analysed in [2] the bilateral version of the game: No edge is built unless both nodes agree on its construction and share the connection cost. This model is better suited for communication network design given that connection costs are typically two sided. They observe that on average more links are created and prove that the worst case price of anarchy of the bilateral setting is worse than for the one-sided model and provide an upper and lower bound.

Anshelevich et al. consider in [1] a network formation game with fair cost allocation, i.e. the cost of each edge is divided equally between the players whose connections make use of it. In their model, nodes pay not only for edges they form an endpoint of, but also edges they use to reach other nodes. Anshelevich et al.'s main interest lies in the quality and the structure of the best Nash equilibrium.

Yet another approach is taken by Eidenbenz et al. in [3], where they study a topology game in wireless communication networks. The agents aim is to adjust their power level to minimize their energy consumption while reaching a transmission range large enough to stay connect with the other agents they wish to communicate. In the paper they give upper and lower bounds on the price of anarchy and present their results on the computational complexity of finding Nash equilibria.

3 Model

Formally, a game in its normal form is defined as the tuple (I, S_i, U_i), where I is the set of players, S_i is the set of strategies for player $i \in I$ and $U_i : \prod_i S_i \to \mathbb{R}$ is the utility function for player $i \in I$.

In our model, we have a finite set of players $I = 0, 1, \ldots, n-1$ denoted by $[n]$, of which each player is associated with a node v. The agents decide to which nodes to build an edge, i.e. they choose a subset of $V\backslash v$. Hence $S_i = 2^{[n]-i}$. The game is fully specified once we define the utility functions. In this report we examine the following game:

THE LOCALITY GAME:
Given a combination of strategies $s = (s_0, \ldots, s_{n-1}) \in S_0 \times \cdots \times S_{n-1}$, we consider the underlying Graph $G(s) = ([n], \cup_{i=0}^{n-1}(i \times s_i))$. In the locality game the cost incurred by each player i under s is defined by

$$c_i = \alpha|s_i| + \sum_{j \neq i} stretch_{G(s)}(i, j),$$

where $stretch_{G(s)}(i, j)$ is the shortest distance in G between i and j divided by the Euclidean distance. Every agent attempts to minimize its cost by building as few edges as possible and yet having short paths to all remaining nodes.

The *social cost* is the sum of each player's cost, which for any situation where no connection is paid for twice is

$$C(G) = \sum_i c_i = \alpha|E| + \sum_{j \neq i} stretch_{G(s)}(i, j).$$

Since we need at least $n-1$ edges to have a connected graph and the stretch between two nodes is at least 1, we obtain the following lower bound for the social cost:

$$C(G) \geq \alpha(n-1) + n(n-1).$$

If the stretch between two nodes is greater than $\alpha + 1$, one of the nodes will construct an edge between them, thus reducing its cost by at least α. Assuming that a node i chooses to pay for k edges, its costs are:

$$\begin{aligned} c_i &\leq \alpha k + (\alpha+1)(n-k) + k \\ &= \alpha(n+1) + k \in O(\alpha(n) + n). \end{aligned}$$

Therefore the social cost is strictly less than $\alpha\binom{n}{2} + \alpha n(n-1)$.

$$\begin{aligned} C(G) &< \alpha\binom{n}{2} + \alpha n(n-1) \\ &\leq \frac{3}{2}\alpha n(n-1) \in O(\alpha n^2). \end{aligned}$$

Unlike the game suggested by Fabrikant et al. where the social optimum for $\alpha > 2$ is always a star regardless of the position of the nodes, our game favours settings with a different connection topology. This has the advantage

that situations where one node carries the burden of forwarding everybody else's messages are much less likely in our model. Moreover it models more closely networks found in reality.
Finding Nash equilibria and computing the price of anarchy is a hard problem. Not even simulating a network formation process is easy, as calculating the best response for a node is NP-complete like in the game presented by Fabrikant et al. Even restricting the game to a one dimensional setting yields non trivial results. In the rest of this thesis, we will therefore focus on the one dimensional case.

4 Undirected Locality Game

In the undirected version of this game, once an edge is built, every player can use it in both directions. Hence only situations where one endpoint is paying for an edge can form a Nash equilibrium.
Allowing undirected links, the lower bound for the social cost is attainable: If we place n nodes along a line and all nodes build an edge to their neighbour on the right, every pair of nodes is connected by a path of exactly the distance between them, hence the stretch is always 1. In this situation no player has an incentive to change its strategy, hence it is also a Nash Equilibrium. Moreover this implies a price of stability of 1, regardless of the choice of α.

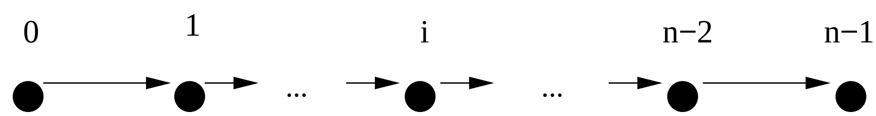

Figure 1: Undirected: Linked list - Social Optimum and Nash equilibrium

Theorem 4.1. *The social optimum in the undirected locality game is a Nash equilibrium, namely the linked list with cost*

$$C(G) = \alpha(n-1) + n(n-1).$$

Under the condition that α is sufficiently large, a star like structure can be a Nash Equilibrium: Consider n Nodes with distance 1 between two neighbouring nodes. If node i builds an edge to all other nodes and no other node decides to add an edge, i cannot change its strategy without being punished to infinite cost. If $\alpha > 2n-3$ no other node will build edges, as the largest stretch between two nodes is always smaller than α and by building an edge no stretch to other nodes is proved. If the node paying for all $n-1$ edges is the leftmost/rightmost node, the inflicted costs are particularly high.

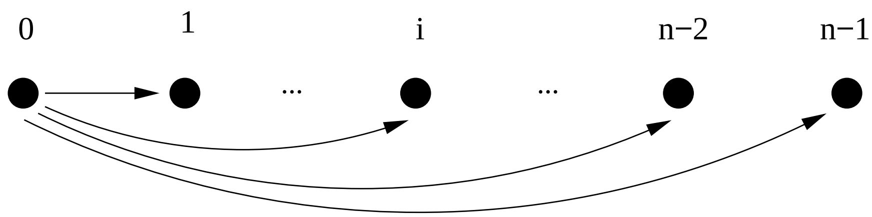

Figure 2: Star

Theorem 4.2. *The price of anarchy of the undirected locality game is in* $\Omega(\log n)$ *for* $\alpha > 2(n-1)$.

Proof. To show this, we compute the social cost of the graph in Figure 2. The cost for the node 0 is $c_0 = \alpha(n-1) + n - 1$. A node at distance i to node 0 pays $\sum_{j=1}^{i} \frac{i+(i-j)}{j}$ for its left neighbours and $\sum_{j=1}^{n-1-i} \frac{i+(i+j)}{j}$ for its right neighbours. This leads to a total sum of

$$\begin{aligned} C(star) &= \alpha(n-1) + n - 1 + \sum_{i=1}^{n-1} \sum_{j=1}^{i} \frac{4i}{j} \\ &= \alpha(n-1) + n - 1 + 4\sum_{i=1}^{n-1} iH_i \\ &= \alpha(n-1) + n - 1 + 4\left(\frac{n(n-1)}{2} H_{n-1} - \frac{(n-1)(n-2)}{4}\right) \\ &= (n-1)(\alpha + 2nH_{n-1} - n + 3). \end{aligned}$$

Thus the price of anarchy is at least

$$\begin{aligned} \frac{C(star)}{C(linked\ list)} &= \frac{(n-1)(\alpha + 2nH_{n-1} - n + 3)}{(n-1)(\alpha + n)} \\ &= \frac{(\alpha + 2nH_{n-1} - n + 3)}{(\alpha + n)} \\ &\geq \frac{(\alpha + 2n\ln(n-2) - n + 3)}{(\alpha + n)} \\ &= \frac{(2n - 3 + 2n\ln(n-2) - n + 3)}{(2n - 3 + n)} \\ &\geq \frac{2}{3}\ln(n-2) \in \Omega(\log n). \end{aligned}$$

□

5 Directed Locality Game

In contrast to the undirected case, it matters which node pays for an edge in the directed version of the locality game. An edge can only be used in one direction: from the node that carries the cost to another node. This game models important characteristics of P2P systems accurately. Agents wish to be highly connected since this improves their searches. At the same time having few neighbours is advantageous to avoid being obliged to forward many searches. Additionally, in P2P networks, user a may know (i.e. have an entry in its routing table) user b, who however may in turn be totally oblivious to the fact that user a even exists. In the directed case of the locality game we encounter a few additional difficulties and finding the social optimum is more intricate. The social cost is

$$C(G) \geq \alpha n + n(n-1).$$

The equivalent to the undirected social optimum, a doubly linked list is no longer always the social optimum in the directed version. It is however always a Nash equilibrium since no node can delete an edge without disconnecting nor will further edges be added as they will not decrease the stretch between any nodes.

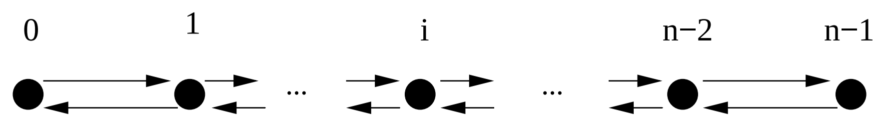

Figure 3: Directed: Doubly linked list

Theorem 5.1. *The price of stability for the directed version of the 1-D locality game is at most 2.*

Proof. The cost for the doubly linked list is $C(G) = 2(n-1)\alpha + n(n-1)$ from which follows a price of stability of less than

$$\frac{2(n-1)\alpha + n(n-1)}{n\alpha + n(n-1)} \leq 2$$

as the cost of the best Nash equilibrium cannot exceed the cost of the doubly linked list.

□

To demonstrate that the doubly linked list is not always the social optimum, it suffices to construct a strategy combination with lower costs which constitutes a Nash equilibrium. Let us consider equidistant nodes with edges from left to right between two neighbouring nodes and an edge from the

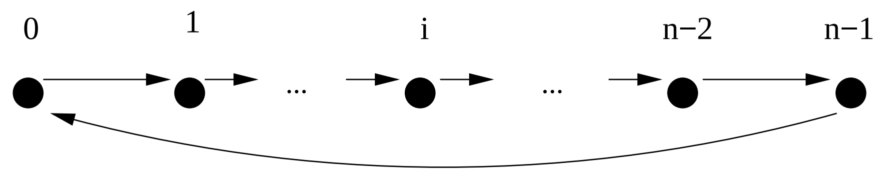

Figure 4: Directed: circular list

rightmost node to the leftmost node. Node 0's cost amounts to $\alpha + n - 1$ whereas node i pays

$$\begin{aligned} c_i &= \alpha + n - 1 - i + \sum_{j=0}^{i-1} \frac{n-1-i+n-1+j}{i-j} \\ &= \alpha + n - 1 - i + \sum_{j=0}^{i-1} \frac{2(n-1)}{i-j} + \frac{j-i}{i-j} \\ &= \alpha + n - 1 - 2i + 2(n-1)H_i. \end{aligned}$$

The social cost for this scenario is

$$\begin{aligned} C(circular\ list) &= n\alpha + n - 1 + \sum_{i=1}^{n-1} n - 1 - 2i + 2(n-1)H_i \\ &= n\alpha + 2n(n-1)H_{n-1} - 2(n-1). \end{aligned}$$

The largest stretch is between node $n-2$ and node $n-1$, namely $2n-3$, so for $\alpha > 2(n-1)$ the circular linked list is an equilibrium.

While the price of stability is bounded by 2, we prove in the following theorem that the price of anarchy is not bounded by a constant, even in the one dimensional line! We saw in Section 3 that the social cost is in $O(\alpha n^2)$. Hence the price of anarchy is in $O(\alpha)$ regardless of the distances between the points and their number. This bound is also tight: There is an instance which has a Nash equilibrium of cost $\Omega(\alpha)$ times the cost of the social optimum.

Theorem 5.2. *The price of anarchy in the directed locality game is* $\Theta(\alpha)$ *for* $2 < \alpha < n$.

Proof. We prove this theorem by constructing a Nash equilibrium that meets this bound.
Let n vertices be placed as illustrated in Figure 5 forming an exponential chain. The distance between the leftmost node and a node with an even

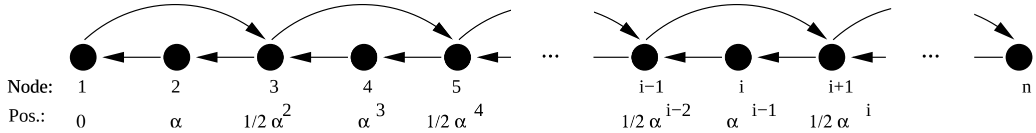

Figure 5: A Nash equilibrium of asymptotically maximal cost

number i is α^{i-1}, for odd numbers $\frac{1}{2}\alpha^{i-1}$. Every odd node i establishes edges to $i-1$ and $i+2$. Even nodes build an edge to their neighbour on the right, $i-1$. Hence we have a linked list from right to left and links from left to right, omitting every other node.

We start by showing that in this scenario no nodes can deviate from its strategy.

Lemma 5.3. *This exponential chain forms a Nash equilibrium for $\alpha > 2$.*

Proof. We observe that no node can remove an edge nor changing one of its edges to point to a node more to the left without disconnecting. Moreover it cannot deviate from its strategy by choosing to link to a node further to the right without increasing stretch. To demonstrate that this graph forms a Nash equilibrium we therefore need to show that no node is able to gain from building an additional edge or replace one of its existing edges.
Case odd nodes: We begin by arguing that no odd node can reduce its cost by adding a link: Establishing an edge to an odd edge is futile as the stretch between two odd nodes is 1 and cannot be improved. A link to an even node $j, j > i$ would only decrease the stretch to the very node and yield a gain of less than

$$\frac{\frac{1}{2}\alpha^{j+1} - \frac{1}{2}\alpha^{i-1} + \frac{1}{2}\alpha^{j+1} - \alpha^{j}}{\alpha^{j} - \frac{1}{2}\alpha^{i-1}} = \frac{\alpha^{j+1} - \alpha^{j} - \frac{1}{2}\alpha^{i-1}}{\alpha^{j} - \frac{1}{2}\alpha^{i-1}} < \alpha,$$

so it would not be built. Replacing the edge to $i+2$ by an edge to the even node $i+3$ does not help to improve the situation of i either: The reduction of the stretch the directly linked node is is less than α whereas the new stretch to $i+1$ amounts to

$$\begin{aligned} \frac{\alpha^{i+2} - \frac{1}{2}\alpha^{i-1} + \alpha^{i+2} - \frac{1}{2}\alpha^{i+1}}{\frac{1}{2}\alpha^{i+1} - \frac{1}{2}\alpha^{i-1}} &= \frac{4\alpha^{i+2} - \alpha^{i+1} + \alpha^{i-1}}{\alpha^{i+1} - \alpha^{i-1}} \\ &> \frac{4\alpha^{i+2} - \alpha^{i+1} + \alpha^{i-1}}{\alpha^{i+1}} \\ &= 4\alpha - 1 - \frac{1}{\alpha} \end{aligned}$$

and cancels the savings. Moreover this edge implies an increased stretch for all the remaining nodes, odd and even. Exchanging the edge to $i+2$ by an edge even further to the right or inserting multiple edges only deteriorates the situation.

CASE EVEN NODES: Even nodes have also no incentive to add any further links: By laying down an edge to its right neighbour, node i shortens the distance to all nodes on the right. The benefit $B_{i,j}$ on the stretch to an odd node $j,\ j > i$ is

$$\begin{aligned} B_{i,j} &= stretch_{old}(i,j) - stretch_{new}(i,j) \\ &= \frac{\alpha^{i-1} - \frac{1}{2}\alpha^{i-2} + \frac{1}{2}\alpha^{j-1} - \frac{1}{2}\alpha^{i-2}}{\frac{1}{2}\alpha^{j-1} - \alpha^{i-1}} - 1 \\ &= \frac{\frac{1}{2}\alpha^{j-1} + \alpha^{i-1} - \alpha^{i-2}}{\frac{1}{2}\alpha^{j-1} - \alpha^{i-1}} - \frac{\frac{1}{2}\alpha^{j-1} - \alpha^{i-1}}{\frac{1}{2}\alpha^{j-1} - \alpha^{i-1}} \\ &= \frac{2\alpha^{i-1} - \alpha^{i-2}}{\frac{1}{2}\alpha^{j-1} - \alpha^{i-1}}. \end{aligned}$$

For an even node j, the savings amount to

$$\begin{aligned} B_{i,j} &= stretch_{old}(i,j) - stretch_{new}(i,j) \\ &= \frac{\alpha^{i-1} - \alpha^{i-2} + \alpha^{j} - \alpha^{j-1}}{\alpha^{j-1} - \alpha^{i-1}} - \frac{\alpha^{j} - \alpha^{i-1} - \alpha^{j-1}}{\alpha^{j-1} - \alpha^{i-1}} \\ &= \frac{2\alpha^{i-1} - \alpha^{i-2}}{\alpha^{j-1} - \alpha^{i-1}}. \end{aligned}$$

As the profit for odd nodes is greater, the savings sum up to strictly less than

$$\begin{aligned} B_i &= \sum_{j>i} B(i,j) \\ &< \sum_{j>i} \frac{2\alpha^{i-1} - \alpha^{i-2}}{\frac{1}{2}\alpha^{j-1} - \alpha^{i-1}} \\ &\leq \sum_{j>i} \frac{4\alpha^{i-1} - 2\alpha^{i-2}}{\alpha^{j-1}} \\ &= 2(2\alpha^{i-1} - \alpha^{i-2}) \sum_{j>i} \frac{1}{\alpha^{j-1}} \\ &= 2(2\alpha^{i-1} - \alpha^{i-2}) \frac{\alpha^{n-i+1} - \alpha^{j-1}}{\alpha^{n+1} - \alpha^{n}} < \alpha \qquad \forall \alpha > 2. \end{aligned}$$

This shows that constructing this edge would be of no avail. Linking to an odd edge $j, j > i + 1$ does not help either as the gain is even smaller.

A link to an even node j would entail a stretch of 1 to the very node instead of

$$stretch(i,j) \quad = \frac{\alpha^j - \alpha^{j-1} + \alpha^{i-1} - \alpha^{i-2}}{\alpha^{j-1} - \alpha^{i-1}} < \alpha.$$

The stretch to all other nodes on the right increases when using the new edge, as the distance from node i to node $i+1$using the new edge is greater than without.

$$\begin{aligned} d &= distance_{new}(i, i+1) - dist_{old}(i, i+1) \\ &= \left(\alpha^{i+1} - \alpha^{i-1} + \alpha^{i+1} - \frac{1}{2}\alpha^i\right) - \left(\alpha^{i-1} - \frac{1}{2}\alpha^{i-2} + \frac{1}{2}\alpha^i - \frac{1}{2}\alpha^{i-2}\right) \\ &= 2\alpha^{i+1} - \left(\alpha^i + 2\alpha^{i-1} - \alpha^{i-2}\right) > 0. \end{aligned}$$

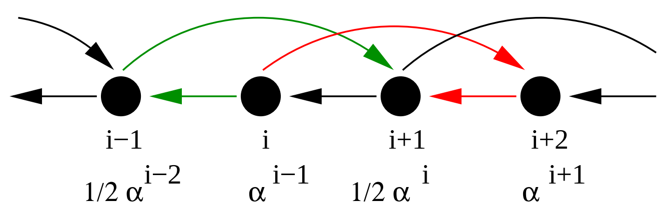

Figure 6: Distances from node i using different paths

which can be seen in Figure 6. Hence there is no gain to other nodes and this edge would not be built. Establishing multiple edges cannot improve the situation either.

Therefore no node can benefit from changing its strategy.

□

Our next step is to examine the total cost of this graph

Lemma 5.4.

The social cost of this exponential chain is in $\Omega(\alpha n + \alpha n^2)$, i.e.

$$C(G) \in \Omega(\alpha n + \alpha n^2).$$

Proof.
There are $n-1$ edges to the left and $\lfloor \frac{n}{2} \rfloor$ edges to the right, aggregating a total cost of $\alpha((n-1) + \lfloor \frac{n}{2} \rfloor)$.

The stretch between two odd nodes is always 1. The stretch between an odd node i and an even node j on the right side of i is

$$\begin{aligned} stretch(i_o, j_e) &= \frac{\alpha^j - \alpha^{j-1} - \frac{1}{2}\alpha^{i-1}}{\alpha^{j-1} - \frac{1}{2}\alpha^{i-1}} \\ &> \frac{\frac{1}{2}\alpha^j - \frac{1}{2}\alpha^{i-1}}{\alpha^{j-1} - \frac{1}{2}\alpha^{i-1}} \\ &> \frac{1}{2}\alpha. \end{aligned}$$

The sum of stretches for an odd node i add up to

$$\begin{aligned} s_{i\ odd} &= \sum_{j<i} stretch(i,j) + \sum_{j>i} stretch(i,j) \\ &> (i-1) + \frac{1}{2}\alpha \left\lfloor \frac{n-i-1}{2} \right\rfloor + \left\lfloor \frac{n-i}{2} \right\rfloor. \end{aligned}$$

The stretch between two even nodes i, j,where $j > i$, is

$$\begin{aligned} stretch(i_e, j_e) &= \frac{\alpha^j - \alpha^{j-1} + \alpha^{i-1} - \alpha^{i-2}}{\geq \alpha^{j-1} - \alpha^{i-1}} \\ &> \frac{\frac{1}{2}\alpha^j - \frac{1}{2}\alpha^{i-1}}{\alpha^{j-1} - \alpha^{i-1}} \\ &> \frac{1}{2}\alpha, \end{aligned}$$

for all $\alpha > 2$.

The stretch between an even node i and an odd node j on the right side is

$$stretch(i_e, j_o) = \frac{\frac{1}{2}\alpha^{j-1} + \alpha^{i-1} - \alpha^{i-2}}{\frac{1}{2}\alpha^{j-1} - \alpha^{i-1}} > 1.$$

Using the above, the sum of stretches for an even node i adds up to

$$\begin{aligned} s_{i\ even} &= \sum_{j<i} stretch(i,j) + \sum_{j>i} stretch(i,j) \\ &> (i-1) + \frac{1}{2}\alpha \left(\left\lfloor \frac{n-i-1}{2} \right\rfloor - 1 \right) + \left\lfloor \frac{n-i-1}{2} \right\rfloor. \end{aligned}$$

Combining the two results, we are now able to compute the sum of the stretches over all nodes:

$$\begin{aligned}
\sum_i s_i &= \sum_{i\ even} s_i + \sum_{i\ odd} s_i \\
&> \sum_i (i-1) + \frac{1}{2}\alpha\left(\left\lfloor\frac{n-i-1}{2}\right\rfloor - 1\right) + \left\lfloor\frac{n-i-1}{2}\right\rfloor \\
&> \frac{n(n-2)}{2} + \frac{\alpha((n-3)(n-2)-n)}{8} + \frac{(n-1)(n-2)}{4} \\
&\in \Omega(\alpha n^2).
\end{aligned}$$

The social cost of this equilibrium is hence

$$\begin{aligned}
C(exp.\ chain) &= \alpha\left(n - 1 + \left\lfloor\frac{n}{2}\right\rfloor\right) + \sum_i s_i \\
&\geq \frac{\alpha(2n-3)}{2} + \Omega(\alpha n^2 + n^2) \\
&\in \Omega(\alpha n^2 + n\alpha).
\end{aligned}$$

□

Now we have all the prerequisites to conclude the proof. The price of anarchy is at least

$$\frac{C(exp.\ chain)}{C(doubly\ linked\ list)} \in \Omega\left(\frac{\alpha n + \alpha n^2}{2\alpha n + n^2}\right)$$

which is in $\Omega(\alpha)$ for all $2 < \alpha < n$. No combination of strategies can involve costs greater than $\alpha\binom{n}{2} + \alpha n(n-1)$, which is in $O(\alpha n^2)$. Hence the above price of anarchy cannot be exceeded.

□

We have proved that the bound on the price of anarchy for $2 < \alpha < n$ is tight, the upper and the lower bound meet asymptotically.
For $\alpha > n$, the social cost of a graph can be at least $\Omega(\log n)$ times worse than the social optimum.

Theorem 5.5. *The price of anarchy for the directed locality game is in* $\Omega(\log n)$.

Proof. Let $\alpha = 2n - 1$. Hence the circular list is a Nash equilibrium and its price of anarchy is at least

$$\begin{aligned}\rho(circular\ list) &= \frac{n\alpha + 2n(n-1)H_{n-1} - 2(n-1)^2}{2(n-1)\alpha + n(n-1)} \\ &\geq \frac{\alpha + 2(n-1)H_{n-1} - 2(n-1)}{2\alpha + (n-1)} \\ &\geq \frac{\alpha + 2(n-1)\ln(n-2) - 2(n-1)}{2\alpha + (n-1)} \\ &\geq \frac{2(n-1)\ln(n-2)}{5n-3} \in \Omega(\log n).\end{aligned}$$

□

In the one dimensional setting every instance has a Nash equilibrium, regardless of the size of α and the distances between the nodes. In higher dimensional spaces however, this does not hold.

Theorem 5.6. *Not every instance of the Locality game has a pure Nash equilibrium.*

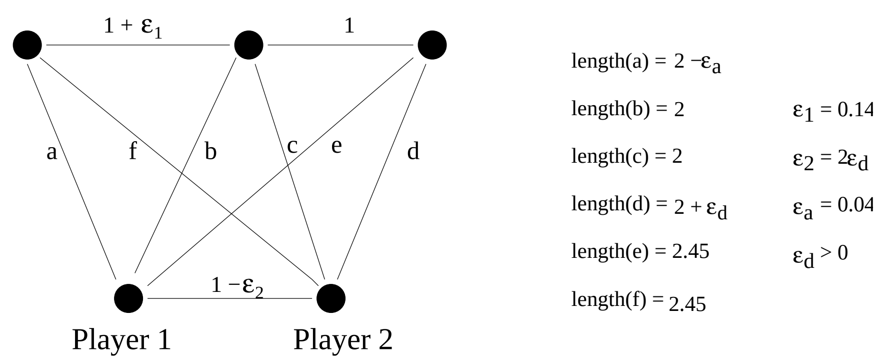

Figure 7: Directed: Instance I has no Nash equilibrium

Proof. We show that the instance of Figure 7 can constitute a situation such that player 1 and 2 will never stop deviating to a better strategy.

Let α be 0.6 and the distances between the nodes as described in Figure 7. First, we observe that in every Nash equilibrium there exists a connection between the nodes in the upper row and the nodes in the lower row, because the stretch would be more than 2, i.e. more than α otherwise.

Player 1 and 2 must have some connection to the nodes in the upper row. We prove, that in all the 2^6 possibilities to do so, there is at least one player that could improve its situation by changing its strategy.

We begin by studying the properties of Nash equilibria of instance I.

Lemma 5.7. *In instance I, Nash equilibria have the following characteristics:*

1. *Neither player 1 nor 2 has three edges to the upper row*
2. *Both players have at least one edge and*
   (a) *Player 1 always establishes edge a*
   (b) *Player 2 does not only build edge f.*

Proof.

1. Player 2 pays at most $2\alpha + 2 + \frac{3+\epsilon_d}{2}$ when establishing the edges c and d, which is less than $3\alpha + 3$, hence building three edges is not worthwhile. Analogously, paying for edges a and b results in costs of at most $2\alpha + 2 + \frac{3}{2.45}$, which is strictly less than $3\alpha + 3$ as well.

2. (a) The cost for player 1 without edge a is

$$\begin{aligned} cost_1(-a) &= c_{2,3} + \frac{3+\epsilon_1}{2-\epsilon_a} \\ &> c_{2,3} + \alpha + 1, \end{aligned}$$

$c_{2,3}$ denoting the cost for connecting to the right and the middle node, whereas the cost of player 1 if it establishes link a is

$$cost_1(a) = c_{2,3} + \alpha + 1,$$

hence building a is always worthwhile.

(b) Assume $c_{1,2}$ is the cost to reach the left and the middle node of the upper row using edges player 1 pays for. When not building any edges, player 2 pays $c_{1,2} + \frac{3.45-\epsilon_2}{2+\epsilon_d}$, which is more than paying for edge d, $c'_{1,2}$ representing the cost for connecting to the left and the middle node being allowed to use edge d as well.

$$\begin{aligned} cost_2(no\ edges) &= c_{1,2} + \frac{3.45-\epsilon_2}{2+\epsilon_d} \\ &> c_{1,2} + \alpha + 1, \\ cost_2(d) &= c'_{1,2} + \alpha + 1, \\ \\ cost_2(f) &\geq \alpha + 1 + c_2 + \frac{3.45-\epsilon_2}{2} \\ &> 2\alpha + 2 + c_2 \\ cost_2(f,d) &= 2\alpha + 2 + c_2. \end{aligned}$$

□

There are 12 remaining possibilities to examine:

Lemma 5.8. *Any pure Nash equilibrium that satisfies the properties of Lemma 2 also complies with the following:*

1. *Player 2 does not link to f*
2. *Player 2 does not build both c and d.*

Proof.

1.

$$\begin{aligned}
cost_2(f,c) &= 2\alpha + 2 + \frac{3}{2+\epsilon_d} \\
cost_2(c) &\leq \alpha + 1 + \frac{3}{2+\epsilon_d} + \frac{3+\epsilon_1}{2.45} \\
&< 2\alpha + 2 + \frac{3}{2+\epsilon_d} \\
cost_2(f,d) &\geq 2\alpha + 2 + \frac{3-\epsilon_2}{2} \\
cost_2(d) &\leq \alpha + 1 + \frac{3-\epsilon_1-\epsilon_a}{2.45} + \frac{3+\epsilon_d}{2} \\
&< 2\alpha + 2 + \frac{3+\epsilon_d}{2}
\end{aligned}$$

2.

$$cost_2(c,d) \geq 2\alpha + 2 + c_1$$

$$cost_2(c) \leq \alpha + 1 + \frac{3}{2+\epsilon_d} + c_1$$

Since d does not help to decrease the stretch to the left-most node, player 2 will only build one edge.

□

There are 6 remaining possibilities to examine, all of which are displayed in Figure 8.

Lemma 5.9. *None of the strategies depicted in Figure 8 forms a pure Nash equilibrium*

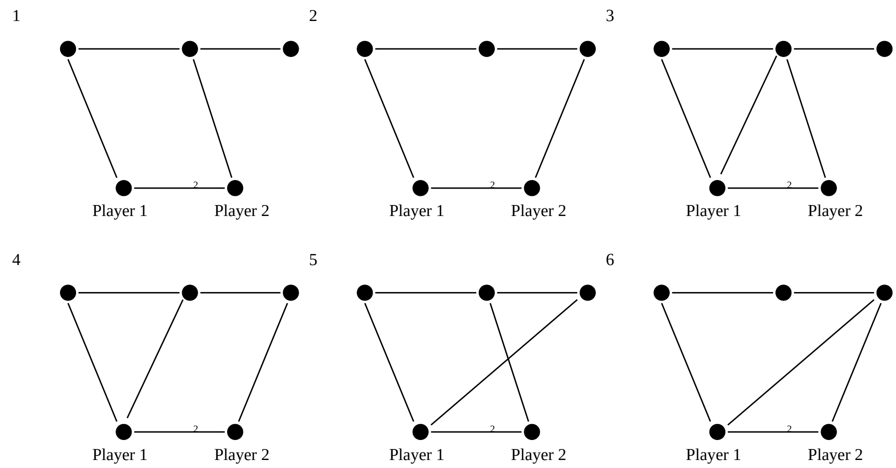

Figure 8: Directed: Remaining candidates for a Nash equilibrium

Proof.
In every scenario there is at least one player that benefits from deviating from its current strategy.

Case 1:

Player 1 adds edge b

$$\begin{aligned} cost_1(-b) &\geq \alpha + 1 + \min(\tfrac{3-\epsilon_2}{2}, \tfrac{3-\epsilon_a+\epsilon_1}{2}) + \min(\tfrac{4-\epsilon_2}{2.45}, \tfrac{4-\epsilon_a+\epsilon_1}{2.45}) \\ &> 2\alpha + 2 + \tfrac{3}{2.45} \\ cost_1(b) &\leq 2\alpha + 2 + \tfrac{3}{2.45} \end{aligned}$$

Case 2:

Player 2 replaces d by c

$$\begin{aligned} cost_2(d) &= c_1 + \alpha + 1 + \tfrac{3-\epsilon_d}{2} \\ cost_2(c) &= c_1' + \alpha + 1 + \tfrac{3}{2+\epsilon_d} \end{aligned}$$

c_1' is the new cost of connecting to the left-most node.
By replacing c by d, this stretch cannot be increased, hence $c_1' \leq c_1$.

Case 3:

Player 2 replaces c by d

$$\begin{aligned} cost_2(c) &= c_1' + \alpha + 1 + \tfrac{3}{2+\epsilon_d} \\ cost_2(d) &= c_1'' + \alpha + 1 + \tfrac{3-\epsilon_2}{2} \end{aligned}$$

Since $\epsilon_2 = 2\epsilon_d$ and $c_1' = c_1''$. it follows that $cost_2(d) < cost_2(c)$.

Case 4,6:

Player 1 removes b

$$\begin{array}{rcl} cost_1(b) & \geq & 2\alpha + 2 + \min(\frac{3}{2.45}, \frac{3-\epsilon_2+\epsilon_d}{2.45}) \\ \\ cost_1(-b) & \leq & \alpha + 1 + \frac{3-\epsilon_a+\epsilon_1}{2} + \frac{3-\epsilon_2+\epsilon_d}{2.45} \\ & < & 2\alpha + 2 + \frac{3-\epsilon_2+\epsilon_d}{2.45} \end{array}$$

Case 5:

Player 1 replaces e by b

$$\begin{array}{rcl} cost_1(e) & \geq & 2\alpha + 2 + \min(\frac{3-\epsilon_2}{2}, \frac{3-\epsilon_2+\epsilon_d}{2.45}) \\ \\ cost_1(b) & \leq & \alpha + 1 + \frac{3}{2.45} \end{array}$$

□

In particular that means that cycles of strategy changes can occur, e.g. $2 \to 1 \to 6 \to 4 \to 2$, see Figure 8. Consequently there exists no combination of strategies that constitutes a Nash equilibrium for this instance.

□

6 Conclusion

We present a network creation game useful in the context of P2P networks. We study the existence and quality of Nash equilibria of this game and prove a tight upper bound on the price of anarchy of $\Theta(\alpha)$. Furthermore we demonstrate that there are instances where no Nash equilibrium esixts.
Some interesting open questions are the following: What is the average cost of a Nash equilibrium? Is there an algorithm that guarantees finding a Nash equilibrium within a constant factor of the social optimum? We are also interested in extending the game to a setting where the network formation is dynamic and on-going.

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