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Distributed Computing

Analysis of Core Constraints and Core Selecting Payment Rules Breaking Non-Decreasing Property

Semester Thesis

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Abstract

Bidders in combinatorial auctions bids for bundle of products. There are various payment methods to decide payments for winners. Some holds non-decreasing property, but some don't. VCG-nearest payment rule is widely used, but its decreasing example is already found. In this paper we will find out necessary conditions for payment decreasing after overbidding by analyzing characteristics of core constraints. To find minimal decreasing example, we will focus on the number of winners and bidders that can not fulfill the conditions to find minimal decreasing example. As it was already proven that auctions with less than 3 winners always holds non-decreasing property, we prove that there can not be decreasing example also in a auction with 3 winners. Next we will show that the number of bidders in the auction have to be bigger than 6.

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Chapter 1

Introduction and Objective

1.1 Combinatorial auction

In combinatorial auction every bidder bids for bundles of goods [1]. If every bidder bids for only one bundle, it is a single minded combinatorial auction. Otherwise, it is a multi-minded CA. In this paper we focus on single minded CA. A single minded CA can be expressed by an undirected graph. Bidders are expressed as nodes and if two bidders have a common item in their bundle, they have an edge between them. To decide on the winners in graph-based representation, we first find a maximum independent sets in the graph and find the independent set that has the largest sum of bids.

Definition 1.1. Single Minded Combinatorial Auction(SMCA): If every bidder in a combinatorial auction bids for one bundle of items, the auction is a single minded combinatorial auction.

1.2 Payment Methods

After deciding winners there are various payment methods to determine payment for each winner. Auction designers want to motivate bidders to bid true value and make enough revenue for sellers at the same time. Truthful bidding implies that if a bidder overbids, its payment should increase or stay the same. This property is called non-decreasing. In some payment methods, non-decreasing property does not hold.

This paper focuses on non-decreasing property of widely used methods VCG and VCG-nearest rule. Winner allocation algorithm $X(b)$ returns allocation x where b is bid profile of all bidders. Social welfare $W(b, X)$ returns $\sum_{i \in x} b_i$.

Definition 1.2. (Non-decreasing Payment Rule) For any allocation x, let B_x be the set of bid profile for which allocation x is efficient. The payment rule p is non-decreasing if for any bidder i, any allcoation x, and bid profile b, $b' \in B_x$ with $b'_i \geq b_i$ and $b'_{-i} = b_{-i}$, $p_i(b', x) \geq p_i(b, x)$ holds [2].

Other than VCG-nearest and VCG payment rule, there are several payment rules. Here we will introduce Proxy and Proportional Payment rules [3].

1.2.1 Proxy Payment

Definition 1.3. Proxy Payment: The proxy payment selects the point in the core where the winners in the auction will share total payment equally. As winners will not pay more than they bid, payment can be expressed as $p_i = min(\alpha, b_i)$ for the minimum $\alpha \geq 0$ such that the point is in the core.

1.2.2 Proportional Payment

Definition 1.4. Proportional Payment: With the proportional payment rule, the winning bidders' payments are given by the point in the minimum revenue core and is of the form $p_i = \alpha \cdot b_i$ where $\alpha \in [0, 1]$.

These two payment rules are proven to be non decreasing in SMCA [4].

1.2.3 VCG Payment

VCG payment[5] evaluates the contribution of the bidder to the auction. It measures the difference of social welfare when the bidder was not in the auction and sum of winning bidders except the bidder.

Definition 1.5. VCG payment: VCG payment of bidder i is determined as below.

$$p_i^{vcg} := W(b, X(b_{-i})) - W(b, x_{-i}) \tag{1.1}$$

As there is no b_i term on the RHS, increasing b_i cannot change her own VCG payment. Other winners' VCG payment will stay the same or decrease. With VCG payment rule, non-decreasing property holds.

Well known method VCG payment rule guarantees bidders to bid truly, but the revenue for the seller is too low [6]. To overcome the short coming of VCG payment rule, VCG-nearest payment rule is devised [7, 8]. This paper finds minimal decreasing example in auctions with VCG-nearest payment rule.

1.2.4 VCG-Nearest Payment

Definition 1.6. VCG-nearest payment method: VCG-nearest payment rule(VN payment rule) select payment point that has the shortest Euclidean distance within minimum revenue core from VCG point. As VN method choose payment point from core, it is one of core-selecting payments rules.

1.3 Core Constraints

Core is a set of payments, for which no coalition is willing to pay more than the winners [8].

Definition 1.7. Core Constraints: A core can be defined by a intersection of a set of inequalities that determine minimum amount of payments for the each set of the bidders. The set of payments have to fulfill following constraint for all possible $L \subset N$ where N is set of all bidders in the auction.

$$\sum_{i \in N \setminus L} p_i(b, x) \geq W(b, X(b_L)) - W(b, x_L) \tag{1.2}$$

$N \setminus L$ can include both winners and losers, but as payment of losers are all 0, core constraints generated by $N \setminus L$ that have loser in it can be covered by other core constraints that only have winners on LHS.

Definition 1.8. Minimum Revenue Core: Minimum revenue core is set of payments in the core that minimize $\sum p_i$

1.3.1 Effective Core Constraints

Some of the core constraints can be covered by other core constraints. Therefore, when we shape core or minimum revenue, we don't have to consider all the core constraints. Minimum revenue core can be decided by core constraints that are not covered by any other core constraints.

Definition 1.9. Effective Core Constraints: If a core constraints can not be covered by other core constraints, it is a effective core constraint that forms core.

In other words, a core defined by the set of all effective core constraints is the same as a core defined considering all the core constraints.

Core constraints can be divided by the number of payments of winners on the LHS. For core constraints that have only 1 winner on the LHS, their RHS value is always the VCG payment of the winner. To denote the divided class of the core constraints, we will define layers of core constraints.

Definition 1.10. i-th layer of core constraints means the set of core constraints that have payments of i bidders on the LHS.

Theorem 1.11. *All winners' payments have to appear at least once in the set of effective core constraints.*

Proof. If core constraints higher than first layer containing bidder k not effective, at least first layer constraint is effective as payment of winner bidder k must be greater than or equal to vcg_k □

Theorem 1.12. *For $L \subset N$ if $k \in X_L$, then $W(b, X(b_L)) - W(b, X(b_{L-\{k\}})) \leq b_k$. If bidder k is one of the winners, the difference between sum of winning bids with bidder k in the auction and not in the auction is less than or equal to b_k.*

Proof. If $L \subset P$, $W(b, X_L) < W(b, X_P)$

$$W(b, X(b_L)) - b_k = W(b, X(b_{L-\{k\}-\{neighbor\ of\ bidder\ k\}})) \leq W(b, X_{L-\{k\}}) \quad (1.3)$$

□

Chapter 2

Related Works

2.1 Conditions for Non-Decreasing

2.1.1 Single Effective Core Constraint

It is already known that if SMCA has single effective core constraint, it always holds non-decreasing property [9]. As effective core constraints have to cover all the winners, if the auction has single effective core constraint it always has form of $p_1 + ...p_n \geq G$ where bidder 1,2,...n are the winners. Minimum revenue is $G = W(b, X(b_{-1,-2...-n}))$. Without loss of generality, let's assume bidder 1 is the overbidder. Let H be the set of all winners and S⊂ H be the set of winners who pay as much as they bid.

$$p_i = min(b_i, \frac{G - \sum_{j \in S} b_j - \sum_{k \in H \setminus S} vcg_k}{n - |S|} + vcg_i) \tag{2.1}$$

After bidder 1 overbids, vcg_1 does not change.

$$p'_i = min(b'_i, \frac{G - \sum_{j \in S'} b_j - \sum_{k \in H \setminus S} vcg'_k}{n - |S'|} + vcg_i) \tag{2.2}$$

vcg payments of other winners except bidder 1 decrease or stay the same, $|S| \geq |S'|,\ S' \subset S$. If $p'_1 == b'_1$, it is obvious p_1 can not decrease. Otherwise, if bidder 1's overbid decreases $\sum_{k \in H \setminus S} vcg_k$, p_1 can only increase. Else if $\sum_{k \in H \setminus S} vcg_k$ does not change, p_1 will also stay the same. Therefore, if SMCA has SECC, it holds non-decreasing property.

2.1.2 Graph containing Decreasing example

Lemma 2.1. *If a CA C corresponding conflict graph G that has subgraph G' with not non-decreasing property, C also does not satisfy non-decreasing property [10].*

Proof. Let's say k' the bid profile of G' is decreasing example. By choosing bids for all bidders in $G \setminus G'$ 0, it can make same condition with k' in G. □

As graphs contains not non-decreasing graph as subset are also not non-decreasing finding minimal example of not non-decreasing graph is meaningful.

2.2 Minimal example

2.2.1 Decreasing example

A decreasing example with 11 bidders and 6 winners exists [11]. This is the minimal decreasing example found until now. Figure 1.1 is the graph-represented

	bundle of interest	bid	VCG	VN	bid'	VCG'	VN'
bidder 1	$\{1\}$	5	2	$37/12$	5	1	$36/12$
bidder 2	$\{2\}$	5	0	$16/12$	5	0	$18/12$
bidder 3	$\{3\}$	4	1	$37/12$	5	1	$36/12$
bidder 4	$\{4\}$	1	0	$7/12$	1	0	$6/12$
bidder 5	$\{5\}$	1	0	$7/12$	1	0	$6/12$
bidder 6	$\{6\}$	1	0	$10/12$	1	0	$12/12$
bidder 7	$\{1,2,4\}$	5			5		
bidder 8	$\{2,3,5\}$	5			5		
bidder 9	$\{1,3,6\}$	7			7		
bidder 10	$\{4,5,6\}$	2			2		
bidder 11	$\{2,3,4\}$	5			5		

Table 2.1: 11 bidders 6 winners decreasing example

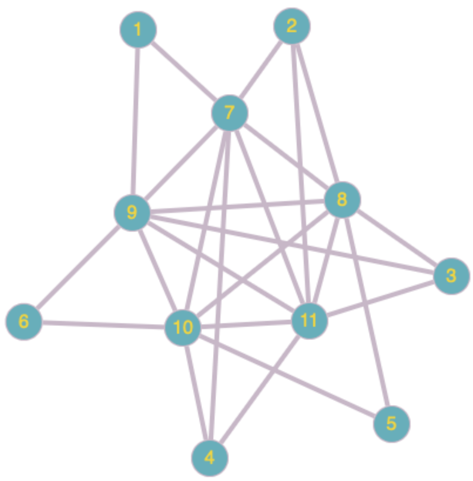

Figure 2.1: 11bidders-6winners decreasing example conflict graph

decreasing example. In this example overbidding of bidder 3 decreased vcg_1. In this example, there are 5 tight core constraints. On this bid profile and interests, the payment point falls on intersection of these 5 equations.

$$p_1 + p_2 + p_4 \geq 5 \tag{2.3}$$

$$p_1 + p_3 + p_6 \geq 7 \tag{2.4}$$

$$p_2 + p_3 + p_4 \geq 5 \tag{2.5}$$

$$p_2 + p_3 + p_5 \geq 5 \tag{2.6}$$

$$p_4 + p_5 + p_6 \geq 2 \tag{2.7}$$

As they are all tight before overbidding, by transforming expressions above we can get following relations:

$$p_1 = p_3,\ p_4 = p_5,\ p_1 = p_4 + 5/2\ p_2 = p_6 + 1/2 \tag{2.8}$$

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 9.5 \tag{2.9}$$

As the auction has 6 winners it can have at most 5 different tight core constraints to have decreasing example. If they have 6, whether vcg point moves or not, all the payments will be fixed. In equation 2.9 we got minimum revenue value by transforming equation 2.3-7, In equation 2.8 we can see that if decrease of vcg_1 results in decreasing of p_1, p_3 will also decrease.

2.2.2 One or two winners

Lemma 2.2. *If SMCA has one or two winner(s), it always holds non-decreasing property.*

Proof. If there is only one winner in SMCA, it has SECC, so it holds non-decreasing property. For 2 winners case, there are 3 core constraints.

$$p_1 \geq vcg_1,\ p_2 \geq vcg_2 \tag{2.10}$$

$$p_1 + p_2 \geq W(b, X(b_{-1,-2}) \tag{2.11}$$

If $vcg_1 + vcg_2 > W(b, X(b_{-1,-2})$, p_1 and p_2 are fixed to vcg_1, vcg_2. As bidder can not change vcg payment of itself and can not pay less than vcg payment, it is non-decreasing. Else, the auction has SECC condition, so it holds non-decreasing property. ☐

Chapter 3

Effectiveness and Tightness of Core constraints

As we saw from the 11 bidders 6 winners example, the decrease of payment after the overbidding can be explained by the effective core constraints that were tight. As overbidding of winning bidder can not change the core, the payment point will move on the intersection of the tight constraints continuously until it meets other effective core constraints. To know the necessary conditions for decreasing example, we need to know how effective core constraints are determined and distinguish subset of effective core constraints that directly affect movement of payment point in given bid profile.

3.1 Tight Core Constraints

Given bidders' interests and bid profile, we can determine the winners and their payments. As VCG-nearest payment rule selects point from minimum revenue core, the payment point falls on intersection of the minimum revenue core and effective core constraint(s).

Definition 3.1. Tight Core Constraints: Effective core constraint(s) where VN point falls, so that sum of payments on its LHS has the same value as its RHS.

For example, if there is an auction with 3 bidders 1,2,3 and 2 product A,B. Their interests and bid profile is like the table 3.1. Winners are bidder 1 and 2. Colored part in Figure 3.1 is the core and $p_1 + p_2 = 7(p_1 \leq b_1, p_2 \leq b_2)$ is minimum revenue core. Core constraint $p_1 + p_2 \geq W(b, X(b_{-1,-2})) = 7$ is effective and tight as it shapes core and VN point falls on the constraint.

[hbt!]

Theorem 3.2. *All of the winners' payments appear at least once in the set of tight core constraints.*

	bidder 1	bidder 2	bidder 3
Interest	A	B	A,B
Bid	4	5	7
VCG	2	3	
VN	3	4	

Table 3.1: 3 bidders 3 products example

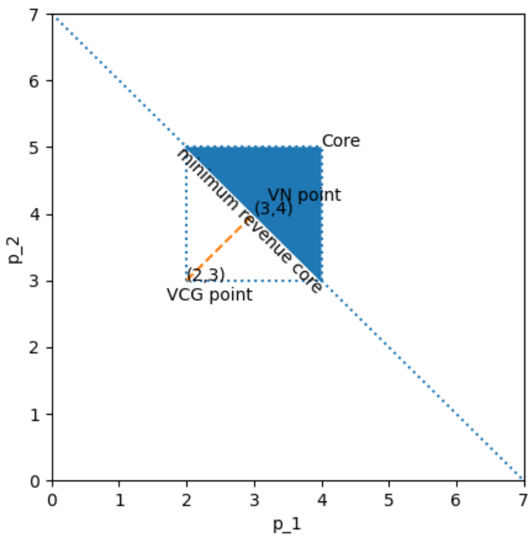

Figure 3.1: Caption

Proof. Let's assume there are n winners and bidder k is one of them. Minimum revenue is m. If all effective core constraints that contain p_k on the LHS are not tight, we can reduce p_k without violating effective core constraints. It contradicts that m is minimum revenue. □

3.2 Covered Core Constraints

In this section we will find necessary conditions for constraints to be effective. To be effective, the constraint should not be able to covered by other constraints.

3.2.1 Covered by Higher Layer

For bidder $k \in x$, if $k \in L$ and $k \notin X_L$

$$\sum_{i \in N \setminus L} p_i \geq W(b, X_L) - W(b, x_L) \tag{3.1}$$

is covered by

$$\sum_{i \in N \setminus \{L-\{k\}\}} p_i \geq W(b, X_{L-\{k\}}) - W(b, x_{L-\{k\}}) \tag{3.2}$$

If $k \in L$ and $k \in X_L$ even if bidder increase her bid, $W(b, X_L) - W(b, x_L)$ does not change. For a core constraint to be effective, $X(b_{...})$ term on the RHS of it have to contain all the winners not on the LHS.

Theorem 3.3. *Winners' overbidding does not change the effective core constraints, so core and minimum revenue also can not be changed.*

Proof. Let's say bidder 1 is one of the winners and overbids. For the core constraints that has p_1 on the LHS, even if b_1 increases, there is no change on the RHS as there is no b_1 term.

For the core constraints that does not have p_1 on LHS, if the core constraint was effective before bidder 1 overbids, $X(b,_{...})$ includes bidder 1, so RHS does not change as b_1 term will be cancelled by $-W(b, x_{...})$. If the core constraint was not effective which means $X(b_{...})$ does not contain bidder 1, when bidder 1 overbids, RHS value decreases until bidder 1 gets included in $X(b_{...})$. After bidder 1 gets in $X(b_{...})$, RHS value is smaller than initial value and can not change anymore, so it can not be effective. Minimum revenue is determined by effective core constraints. Therefore, winning bidders overbidding can not change effective core constraints, core and minimum revenue. □

To change VN-point, the only way is to move VCG point.

Theorem 3.4. *If a winner $k \in L$ and $k \notin X_L$ before overbidding, $\sum_{i \in N \setminus L} p_i \geq W(b, X_L) - W(b, x_L)$ can not be effective no matter how much bidder k overbids.*

Proof. Bidder k can be included in X_L after overbidding. At the point bidder k starts to be included in $X_L W'(b, X_L) - W'(b, x_L) = W(b, X_L) - W(b, x_L)$ and from that point, even if bidder k overbid more, $W(b, X_L) - W(b, x_L)$ does not change. □

3.2.2 Covered by Lower Layer

On the example above if $W(b, X_L) - W(b, x_L)$ is equal to $W(b, X_{L-\{k\}}) - W(b, x_{L-\{k\}})$, the lower layer will cover the higher layer's constraint as winners' payments are at least 0.

3.2.3 The First layer

if $|N \setminus L| = 1 \rightarrow$ *RHS of the constraints* $=$ *VCG payment.* In VN payment rule, this core constraints can be always fulfilled.

Chapter 4

Number of Winners and Bidders

It is already proven that CAs with 1 or 2 winners can not have decreasing example. This chapter will prove that CAs with 3 winners always holds non-decreasing property and find conditions for decreasing example with 4 winners CAs.

4.1 3 Winners Case

For 3 winner CAs, their core can be presented in 3-dimensional graph. Core constraints are expressed as planes. Without loss of generality let's say bidder 1,2,3 are winners and bidder 1 is overbidder. There are 2 big cases for 3 winners. One is when minimum revenue is $W(b, X_{-1,-2,-3})$ and the other case is when the 3 plane of 2-nd layer constraints meets in one point. In the latter case, the point is always the VN point, so no matter one winner overbids or not payment of all bidders are fixed. non-decreasing property always holds.

In the former case, it can be again divided to 3 cases (before overbidding) 1) VN point falls on the plane of minimum revenue core ($p_1+p_2+p_3 \geq W(b, X(b_{-1,-2,-3}))$), 2) VN points fall on the intersection of the 3rd layer core constraint and a 2nd layer core constraint formed by bidder 2,3 who are not overbidding and 3) VN point falls on the intersection of the 3rd layer core and a 2nd layer core constraint formed by 2 winners including bidder 1. Let's think of any auction that has 3 winners bidder 1, 2, 3 with their bid b_1, b_2, b_3 and their VCG payment vcg_1, vcg_2, vcg_3. Without loss of generality, let bidder 1 overbids.

$$p_1 + p_2 + p_3 \geq W(b, X_{-1,-2,-3}) \tag{4.1}$$

$$p_1 + p_2 \geq W(b, X_{-1,-2}) - b_3 \tag{4.2}$$

$$p_2 + p_3 \geq W(b, X_{-2,-3}) - b_1 \tag{4.3}$$

$$p_1 + p_3 \geq W(b, X_{-1,-3}) - b_2 \tag{4.4}$$

Let M be the minimum revenue. M is $W(b, X_{-1,-2,-3})$ except the case when constraint 4.2, 4.3, 4.4 covers constraint 4.1 which implies $bidder1 \in X(b_{-2,-3}) \& bidder2 \in X(b_{-1,-3}) \& bidder3 \in X(b_{-1,-2})$. In this case payment of bidder 1,2,3 is fixed to the point $p_1 + p_2 = W(b, X_{-1,-2}) - b_3, p_2 + p_3 = W(b, X_{-2,-3}) - b_1, p_1 + p_3 = W(b, X_{-1,-3}) - b_2$ meets. According to Theorem 3.3, the point will not move after winning bidders overbid.

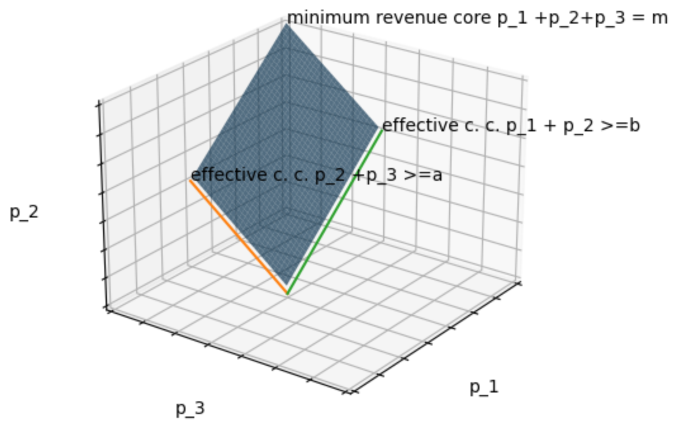

Figure 4.1: example core for 3 winners auction

4.1.1 A line with directional vector (1,1,1) passing VCG point meets minimum revenue core

This is when VN point falls on blue plane in Figure 4.1. Payment (p_1, p_2, p_3) can be expressed as following.

$$(vcg_1 + \frac{M - (vcg_1 + vcg_2 + vcg_3)}{3}, vcg_2 + \frac{M - (vcg_1 + vcg_2 + vcg_3)}{3}, vcg_3 + \frac{M - (vcg_1 + vcg_2 + vcg_3)}{3}) \tag{4.5}$$

Let's see what happens if b_1 increases. $W(b, X_{-1,-2,-3})$ does not change, vcg_2, vcg_3 stay the same or decrease.

$$p\prime_1 = vcg_1 + \frac{M - (vcg\prime_1 + vcg\prime_2 + vcg_3)}{3} \geq p_1 \therefore non-decreasing \tag{4.6}$$

If constraint 4.3 is effective($1 \in X_{-2,-3}$) constraint 4.1 and 4.3 forms a line

$$p_1 = b_1 + M - W(b, X_{-2,-3}) \& p_2 + p_3 = W(b, X_{-2,-3}) - b_1 \tag{4.7}$$

If constraint 4.2 is effective ($3 \in X_{-1,-2}$) constraint 4.1 and 4.2 forms a line

$$p_3 = b_3 + M - W(b, X_{-1,-2}) \& p_1 + p_2 = W(b, X_{-1,-2}) - b_3 \tag{4.8}$$

4.1.2 VN-point on line 4.7

This is when VN point falls on orange line in Figure 4.1. If $bidder1 \notin X_{-2,-3}$ before and after overbidding, constraint 4.3 is not effective, so it does not form effective equation 4.7. Else if $bidder1 \in X_{-2,-3}$ before and after overbidding , if vcg point falls on line 4.7 if $vcg_1 - 1/2(vcg_2 + vcg_3) \geq b_1 + M - W(b, X_{-2,-3}) - 1/2(W(b, X_{-2,-3}) - b_1)$ and p_1 is fixed to $b_1 + M - W(b, X_{-2,-3})$. Even if vcg_2 *or* vcg_3 decreases we can see that VN point will fall on the line 4.7.

Else there is a case of $bidder1 \notin X_{-2,-3}$ before overbidding and $bidder1 \in X_{-2,-3}$ after overbidding. According to theorem 3.4, after overbidding $W'(b, X_L) - W'(b, x_L) = W(b, X_L) - W(b, x_L)$. Then the constraint does not change anymore, so it is the same case with $bidder1 \notin X_{-2-3}$ before and after overbidding. Therefore, once VN point falls on line 4.7, no matter how much bidder 1 overbids, p_1 is fixed. $\therefore$ non-decreasing

4.1.3 VN-point on line 4.8

This is when VN point falls on green line in Figure 4.1. If $vcg_3 - 1/2(vcg_1 + vcg_2) \geq b_3 + M - W(b, X_{-1,-2}) - 1/2(W(b, X_{-1,-2}) - b_3)$, VN point falls on the line 4.8. As VN point moves continuously if the VN point moves to minimum revenue core plane, p_1 can not decrease because it will move to the case of section 4.1.1. Therefore, we have to consider only the case that VN point falls on the line 4.8 before and after overbidding.

The line passing (vcg_1, vcg_2, vcg_3) and VCG-nearest point is perpendicular with line 4.8 which has direction vector (1,-1,0). Therefore, $p_2 - vcg_2 = p_1 - vcg_2$. $p_1 + p_2 = W(b, X_{-1,-2}) - b_3$. If b_3 increases, vcg_2 decreases or stay the same and vcg_3 doesn't change. Easily can see that p_3 can only increase or stay the same. It will be same with the another line formed by constraint 4.4 and minimum revenue core. $\therefore$ non-decreasing

If a CA has 3 winners, it can not have a decreasing example.

4.2 4 Winners Case

If VN payment falls on the plane of minimum revenue core, it is same with SECC case, so there can not be decreasing example. To have decreasing example, VN payment have to fall on intersection of minimum revenue core and core constaint(s). Let's say minimum revenue core is $p_1 + p_2 + p_3 + p_4 = M$. Without loss of generality, bidder 1 is overbidder and vcg_3 decreases because of bidder 1's overbid.

If 3rd layer of effective core constraint(s) is tight, as tight core constraints have to cover all winners, payments of at least 2 bidders are fixed. Even if p_1 and p_3

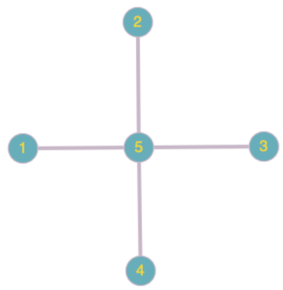

Figure 4.2: 5 bidders 4 winners graph

are not fixed, $p_1 + p_3$ is fixed and decrease of vcg_3 can not decrease payment of bidder 1.

4.2.1 Necessary Conditions for Decreasing example

The only cases that can have decreasing example with 4 winners is when $p_1+p_2 = a, p_2 + p_3 = b$ are tight and additionally one of $p_1 + p_4 = c, p_3 + p_4 = d$ is tight. If $p_1 + p_3 = e \; or \; p_2 + p_4 = f$ is tight, decrease of vcg_3 can not make p_1 smaller.

To make $vcg_3 = W(b, X(b_{-3})) - b_1 - b_2 - b_4$ decrease after b_1 increases, $X(b_{-3}) \not\ni 1 \; and \; X(b_{-3}) \ni (bidder\ 2$[1] and at least one of bidder 3's neighbor.[2]

$p_1 + p_2 \geq W(b, X(b_{-1,-2})) - b_3 - b_4$ and $p_2 + p_3 \geq W(b, X(b_{-2,-3})) - b_1 - b_4$ must be effective and their RHS should be bigger than 0. There is loser node who does not neighbor to 3,4. Same for 1 and 4.

$$X(b_{-1,-2}) \ni 3, 4 \; , X(b_{-2,-3}) \ni 1, 4 \tag{4.9}$$

$$X(b_{-3}) \ni 2, \; at\ least\ one\ of\ bidder\ 3's\ neighbor, \; X(b_{-3}) \not\ni 1 \tag{4.10}$$

$$W(b, X(b_{-1,-2}) - b_3 - b_4 > 0, \; W(b, X(b_{-2,-3}) - b_1 - b_4 > 0 \tag{4.11}$$

4.11 means there is a node that does not neighbor to node 3 and 4. Same for node 1 and 4. Node 3 must have neighbor node not adjacent to node 2.

4.2.2 5 Bidders

The only possible case for 5 bidder auction to have 4 winners is like the Figure 4.1. It is complete bipartite graph, so it has SECC [12]. Therefore, there is no decreasing example in auctions with 5 bidders.

[1] If X_{-3} does not contain *bidder* 2,$X_{-3} = X_{-2,-3}$, so it contains bidder 1, so bidder 1's overbidding can not decrease vcg_3.

[2] Otherwise, bidder 1 can't be winner

4.2.3 6 Bidders

To be tight, $p_1 + p_2 \geq a$ and $p_2 + p_3 \geq b$have to be effective first. $X_{-1,-2} \ni 3, 4$ and $X(b_{-2,-3}) \ni 1, 4$. Overbidding of bidder 1 have to decrease vcg_3. Which means $X_{-3} \not\ni 1$ and $X_{-3} \ni 2$. Only graph in Figure 4.2 fulfills these conditions. If node 2 and 6 is connected, $X(b_{-3})$ can not contain both node 2 and 6.

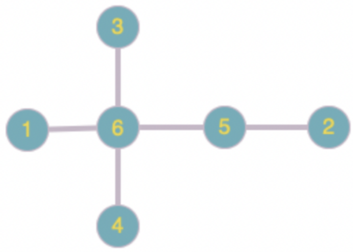

Figure 4.3: 6 bidders 4 winners graph

If there are 6 bidders and 4 winners, the node that does not neighbor to node 3,4 also does not neighbor to node 1 and 4. Because node 4 should be connected to the graph. Let's say the node is node 5.Then node 1,3,4 neighbor to node 6. $X_{-1,-2} = \{3, 4, 5\}$, $X_{-2,-3} = \{1, 4, 5\}$ $X_{-3} = \{2, 6\}$.

$$vcg_2 = W(b, X_{-2}) - b_1 - b_3 - b_4 = b_1 + b_3 + b_4 + b_5 - b_1 - b_3 - b_4 = b_5 \quad (4.12)$$

$a = b = b_5$. Every winner has to pay at least vcg payment of themselves.If $p_1 + p_2 \geq a$is effective and tight, p_1 is 0 and $p_2 = b_5$ [3]. $\therefore$ there is no decreasing example in 6 bidder CAs.

According to theorem 3.4, tight core constraints have to cover all winners. Beside $p_1+p_2 \geq a$ and $p_2+p_3 \geq b$, there should be one more tight core constraint that has p_4 on the LHS. Third layer constraints can not be tight as it will fix one of the payment values and if first layer constraint is tight, it fixes p_4. Only $p_2 + p_3 \geq c$ or $p_1 + p_2 \geq d$ can be tight [4].

4.2.4 $p_3 + p_4 \geq W(b, X(b_{-3,-4})) - b_1 - b_2$ tight case

In addition to $p_1+p_2 \geq W(b, X(b_{-1,-2}))-b_3-b_4$ and $p_2+p_3 \geq W(b, X(b_{-2,-3}))-b_1-b_4$, if $p_3+p_4 \geq W(b, X(b_{-3,-4}))-b_1-b_2$ are tight, some conditions are added. Minimum revenue is $W(b, X(b_{-3,-4})) + W(b, X(b_{-1,-2})) - b_1 - b_2 - b_3 - b_4$.

$$X(b_{-3,-4}) \ni 1, 2, \;\; X(b_{-3}) \ni 4 \quad (4.13)$$

[3] $b_2 > b_5$ as bidder 2 is winner instead of bidder 5.

[4] if $p_2 + p_4 \geq a$ or $p_1 + p_3 \geq b$ is tight, decrease of vcg_3 can't decrease p_1.

There must be a node that does not neighbor to both node 3 and 4. If $X(b_{-3}) \not\ni 4$, $X(b_{-3,-4}) == X(b_{-3})$, so $X(b_{-3})$ should include bidder 4. As vcg_3 is bigger than 0, the graph needs a node that does not neighbor to node 2 and 4.

$$W(b, X(b_{-1,-2}))+W(b, X(b_{-3,-4})) > W(b, X(b_{-1,-3}))+W(b, X(b_{-2,-4})) \quad (4.14)$$

Otherwise, $p_1 + p_4$ and $p_2 + p_4$ will be fixed, so it deviates assumption.

4.2.5 $p_1 + p_4 \geq W(b, X(b_{-1,-4})) - b_2 - b_3$ tight case

In addition to $p_1+p_2 \geq W(b, X(b_{-1,-2}))-b_3-b_4$ and $p_2+p_3 \geq W(b, X(b_{-2,-3}))-b_1 - b_4$, if $p_1 + p_4 \geq W(b, X(b_{-1,-4})) - b_2 - b_3$ are tight, similarly with section above, some conditions can be added. Minimum revenue is $W(b, X(b_{-1,-4})) + W(b, X(b_{-2,-3})) - b_1 - b_2 - b_3 - b_4$.

$$X(b_{-1,-4}) \ni 2, 3 \quad (4.15)$$

$$W(b, X(b_{-1,-4}))+W(b, X(b_{-2,-3})) > W(b, X(b_{-1,-3}))+W(b, X(b_{-2,-4})) \quad (4.16)$$

4.3 7 bidders 4 winners

Possible decreasing examples in an auction with 7 bidders and 4 winners can be organized as the table 4.1 and 4.2. For case 1 in table 4.1, node 1 and 4 share their only neighbor node 7. As $X(b_{-3}) \ni 4$ and $X(b_{-3}) \not\ni$ neighbor of node 1, case 1 can not have decreasing example.

$p_3 + p_4 = W(b, X(b_{-3,-4})) - b_1 - b_2$				
does not neigbor to	Case 1	Case 2	Case 3	Case 4
3,4	5	5	5	5
1,4	5	5	6	6
1,2	6	6	6	6
2,4	6	7	5	6

Table 4.1: 7 bidders 4 winners auction decreasing possible cases

$p_1 + p_4 = W(b, X(b_{-1,-4})) - b_2 - b_3$			
does not neigbor to	Case 1	Case 2	Case 3
3,4	5	5	5
1,4	5	6	6
2,3	6	5	7

Table 4.2: 7 bidders 4 winners auction decreasing possible cases

4.4 Future Work

There is no decreasing example found in 7 bidder 4 winners auctions yet. Minimum number of winners for decreasing example might be bigger than 4. However, it is obvious that a winner's overbidding have to decrease other winner's VCG payment and the VN point has to fall on intersection of minimum revenue core and other effective core constraints to decrease the overbidder's payment. With necessary conditions found in this paper, it will be possible to find minimal decreasing example.

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