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Semester Thesis

Virus Inoculation on Social Graphs - The Friendship Factor

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Abstract

A fundamental assumption in game theory is that the agents participating in a game are selfish. This assumption is reasonable, but sometimes too restricted. In this semester thesis we loosen it to some extent, letting the players still be selfish, but also caring to some degree for other agents. Concretely, we introduce friendship between neighbors while looking at a virus inoculation game which is played on a network of connected agents.

We prove that introducing this social aspect never impairs the community with respect to the total social cost, although social welfare does not monotonously increase with the strength of friendship. We further quantify the social effect in two special topologies, namely in the clique and in the star graph.

1 Introduction

Over the past years, the Internet, as the probably best known network at all, has evolved enormously and became a great platform for a continuously raising number of users and innumerable applications. Unfortunately, not just the good parts of the Internet have made progress, the process has, among others, also led to a huge number of computer-viruses, acting in many different manners and aiming at diverse objectives. However, it is interesting that most of them share a common technique to propagate. Namely, they often try to proceed to those participants of the network who seem to be on friendly terms with the infected computer. For example, they automatically forward themselves to those email-addresses that are stored in the infected computer's address book, so they are actually spreading in *social networks*.

A contaminated computer hence serves as a source for the virus to infect every other directly linked computer, where 'directly linked' has to be understood not as physically but rather semantically connected. Of course, this is not the only effect for a successfully attacked victim. Depending on the nature of the virus, every computer that eventually is infected has also to face a drawback in terms of e.g. lost data or recovering costs.

However, owners of computers have a possibility to avoid an infection by installing anti-virus software. Clearly, such an anti-virus software also needs some investment, may this be money to buy it or time to install it.

A person's decision whether to install anti-virus software on his machine or to renounce it, is of course influenced by the costs of the respective investments. As a first thought one could decide to buy the protection if and only if the expected loss by infection is greater than the cost of the software. However, thinking twice, such a totally selfish decision is too shortsighted. It seems reasonable that installing the software on just a few central devices can limit the risk of an infection to such an extent that this solution is cheaper regarding the whole society.

Considering the total social cost of a network when it is centrally organized compared to when there are only selfish players is an especially interesting question, not only for the just described *Virus Inoculation Game* but generally in game theory. This ratio is known under the term of *Price of Anarchy*.

In this semester thesis we are interested in another quantity, namely the effect to the total social cost when the players are still selfish but now also care to some degree about other agents, namely their neighbors, in the network. We call this ratio the *Windfall of Friendship*.

We believe that introducing a social aspect to networks which are threatened to be polluted by a virus is of great interest for a large number of applications as social networks have become an important topic in various fields today. Linked to computer science we would like to mention additionally the telephone-graph, where participants have stored the numbers of and hence are connected to their friends, as well as the fast raising number of community websites, such as ‘MySpace’[1] or ‘Orkut’[2].

After presenting some related work (in Section 2), we formally describe the Virus Inoculation Game and give some important definitions and observations (in Section 3). Having defined the basis, we first show (in Section 4) that introducing friendship never impairs the community with respect to its total social cost. However, we also give an example to show that stronger friendship does not always implicate lower cost. We then prove that computing the best and the worst equilibrium is NP-complete for any degree of friendship.

Following these general findings, we then investigate (in Section 5) the complete graph to establish that there are problem instances where the worst equilibrium in the social adaption is a factor 4/3 better than the worst equilibrium in the original game and that this is even the best possible improvement. For the star graph, the second special topology we analyze in detail, we find that there are instances for which the worst equilibrium totally disappears thanks to friendship (i.e. there is only the best equilibrium left) and give a condition to check whether this is the case for a given instance of interest. We also state an upper bound for the improvement that can be reached in the star graph.

The thesis is completed (with Section 6) by stating concluding comments and by indicating open problems.

2 Related work

2.1 Social networks

Social networks are structures of linked-up participants which share some common interest. As there is a broad variety of naturally-occurring social networks, consider e.g. kinship or trade, it is not surprising that they have been studied for a

[1]http://www.myspace.com/
[2]http://www.orkut.com/

long time, primarily by sociologists. Recently, Kleinberg introduced the algorithmic study of social networks, causing them to be a currently intensively studied topic in computer science too. Among others, research includes the so called small-world phenomenon [11, 22], the analysis of networks of strategic agents [7, 17, 20] as well as very recent approaches to examine large-scale community websites [13] such as e.g. 'Facebook'[3].

Relations in social networks do not necessarily have to be symmetric. However, in this semester thesis we require the friendship relation to have this property. Furthermore, we do not grade friendship, i.e. two agents are either friends or they are not.

2.2 Game theory

Game theory studies systems of agents which are interacting with each others. The outcome of such a system then depends on the strategy of these agents. This field of research is of great interest in various social sciences and economics. Papadimitriou's statement in [17] that the Internet has a complexity which requires techniques from game theory [16] to be understood resulted in game theory to become very popular in computer science as well.

In the context of selfish agents, there has been a lot of research concerning the Price of Anarchy [12, 19], which has been examined in different systems, such as the Internet [4], wireless ad-hoc networks [3] or peer-to-peer networks [14]. We investigate another quantity in this semester thesis, namely the Windfall of Friendship, which is a measure for the improvement for the community when adding a social aspect.

2.3 Virus games

Traditional models characterize virus infections in terms of birth and death rate of the virus [2, 6]. Since then, the model has been transfered to a directed random graph [8] and later to other kinds of graphs [9, 10, 21] such as Internet-like power-law graphs [18, 23]. Furthermore, [5] attends to malware propagation in mobile phone networks by evaluating realistic simulations. The model used in this thesis has been presented in [1] and does, considering a general undirected graph, not restrict the network topology.

[3] http://www.facebook.com/

The original Virus Inoculation Game can of course be extended in different ways. One recent approach, namely adding Byzantine players, and its according results have been given in [15]. In this semester thesis we go into the opposite direction: Instead of making the environment more vicious, we loosen the selfishness of the players, which results in a friendlier setting.

3 Model

The game we are considering in this semester thesis is the Virus Inoculation Game from [1]. We give its definition and our adaption to it in the following two subsections. However, we first have to give the definition of a concept that is of great importance in general game theory and therefore also for this semester thesis.

Definition 3.1 (Nash Equilibrium)**.** Assume that every player in a given game has chosen his probability distribution over the possible strategies. If no player can, supported by the knowledge of the choices of all the other players, increase his own profit (or decrease his own cost, respectively) by changing his decision, then the system is in a *Nash equilibrium*.

In this semester thesis we restrict ourselves to pure strategies, i.e. every player deterministically chooses one strategy. Accordingly, the system is in a *pure Nash equilibrium* if no player has an incentive to change his strategy.

3.1 Virus Inoculation Game

Having already described the Virus Inoculation Game in the introduction in Section 1, we now express those considerations more formally. We are given players p_i for $i = 1, \dots, n$ that are connected by m links, i.e. they form a graph with n nodes and m edges. Each one of these players is the initial attack-point of the virus with probability $\mathbf{Pr}[p_i \text{ is the infection-source}] = 1/n$.

Every player has the choice between buying an anti-virus software for the price of C or facing the risk of being infected, yielding a damage of cost L. We introduce a variable a_{p_i} for every player p_i that indicates p_i's chosen option. Namely, $a_{p_i} = 1$ means that the player p_i is protected whereas a player p_j that is willing to take the risk sets $a_{p_j} = 0$. The collectivity of these choices is summarized as the *strategy profile* $\vec{a} = (a_{p_1}, \dots, a_{p_n})$.

Note that, as announced before, we are only interested in pure strategies here. Therefore, $a_{p_i} \in \{0, 1\}$ for every player p_i.

The virus will start at one player and not only infect this one (if it is not inoculated) but also all the players that are linked with it over a path of insecure nodes. Hence, the more insecure players are connected, the more dangerous it gets for every one of them. The vulnerable region in which a player p_i lies is denoted as p_i's *attack component*. Note that the attack component is only determined by the strategies of p_i's surrounding nodes but is independent of the strategy of p_i itself.

Sticking it together, we are now ready to give the definition of the actual cost of a player:

Definition 3.2 (Actual Cost)**.** The actual cost of a player p_i is defined as

$$c_a(i, \vec{a}) = a_{p_i} \cdot C + (1 - a_{p_i}) \cdot L \cdot \frac{k_i}{n}$$

where k_i denotes the size of p_i's attack component.

It is natural that the total social cost of a network N, a value that we will particularly be interested in, is defined as the sum of the costs of its players, i.e. $Cost(N, \vec{a}) = \sum_{p_i \in P_N} c_a(i, \vec{a})$ where P_N is the set of players in the network N.

As already mentioned before, a widely studied quantity in game theory compares the total social cost of a network when it consists only of totally selfish players to when it is centrally organized. This quantity is defined as follows:

Definition 3.3 (Price of Anarchy)**.** The Price of Anarchy $PoA(I)$ for an instance I is defined as

$$PoA(I) = \frac{Cost(I, \vec{a}_{worstNE})}{Cost(I, \vec{a}_{OPT})}.$$

To avoid two uninteresting cases, we further need some conditions on C and L. The first case occurs when the cost of the anti-virus software is very high, resulting in a totally insecure network in which no player will inoculate and therefore all nodes eventually will be infected. The other case is the opposite phenomenon. Namely, even with an attack component of size only 1 the expected loss due to infection is higher than the investment of buying the software. This would result in a network in which every player would protect himself and hence no one would ever be infected.

We give the formal correspondence of these considerations in the following observation:

Observation 3.4. In every setting it must hold that

- $C < L$ (to avoid total insecurity)
- $C > \frac{L}{n}$ (to avoid total security)

3.2 Social Aspect

We now introduce the social aspect in the form of the *Friendship Factor* F which describes how much a player will weigh the situation of his neighbors. As $F = 0$ would drive us to the original game as described before and $F \geq 1$ would in some sense violate the selfishness of the players, we ask F to be a value strictly between 0 and 1 when speaking about the social case.

As we will often compare the original game with its social adaption, we sometimes use the notation of *Nash equilibrium* (*NE*) if $F = 0$ and, respectively, *friendship Nash equilibrium* (*FNE*) if $F > 0$.

The decision of a player to buy the protection or not to buy it will now be based on a different cost-function that also takes the costs of his neighbors into account.

Definition 3.5 (Perceived Cost)**.** The perceived cost of a player p_i is defined as

$$c_p(i, \vec{a}) = c_a(i, \vec{a}) + F \cdot \sum_{p_j \in S_{p_i}} c_a(j, \vec{a})$$

where S_{p_i} denotes the neighborhood of p_i.

Note that the perceived cost of a player p_i is only responsible for p_i's selection of one of his options. Nevertheless, in the end p_i will only have to pay $c_a(i, \vec{a})$ as before. Otherwise we would have to adapt the total cost of the network as well, resulting in an impossibility to compare the results in the original game with the one in this social adaption.

However, this comparison is exactly what we are mainly interested in here. To qantify the improvement (or impairment, respectively) that is caused by the addition of the social aspect, we introduce the *Windfall of Friendship*.

Definition 3.6 (Windfall of Friendship)**.** The *Windfall of Friendship* $WoF(I, F)$ for an instance I is the ratio of the total cost of the most expensive Nash equilibrium in the original game over the total cost of the most expensive Nash equilibrium in the social adaption, i.e.

$$WoF(I, F) = \frac{Cost(I, \vec{a}_{worstNE})}{Cost(I, \vec{a}_{worstFNE})}.$$

According to this definition it is clear that a Windfall of Friendship $WoF(I, F) > 1$ is synonymical to an improvement we get by adding the social aspect whereas $WoF(I, F) < 1$ is equivalent to an induced impairment. If $WoF(I, F) = 1$ we neither win nor lose anything.

4 General Results

It has been proven in [1] that in the original game any attack component has size at most Cn/L. We will show in this section that there is a similar characteristic in our social adaption. As every node cares about its neighbors, it seems reasonable to assume that the maximal size of the attack component of a player p_i is dependent on the number of secure and insecure neighbors this player has, and thus differs from player to player.

For the rest of this section, let n denote the number of players and let p_i be one of those. We denote by $S^1_{p_i}$ (with $s^1_{p_i} := |S^1_{p_i}|$) the set of secure neighbors of p_i and by $S^0_{p_i}$ (with $s^0_{p_i} := |S^0_{p_i}|$) the set of his insecure neighbors, respectively. Furthermore, the size of the attack component of p_i is denoted by k_i.

Theorem 4.1. The player p_i will inoculate if and only if the size of his attack component is

$$k_i > \frac{Cn/L + F \cdot \sum_{p_j \in S^0_{p_i}} k_j}{1 + F s^0_{p_i}}.$$

Proof of Theorem 4.1. As, like every player, p_i is selfish, it is clear that he will buy the anti-virus software if and only if this choice is cheaper for p_i. We therefore have to compare the two expected costs of buying it and facing the risk respectively. By Definition 3.5 these costs are

$$c^1_p(i, \vec{a}) = C + F\left(s^1_{p_i} C + \sum_{p_j \in S^0_{p_i}} L\frac{k_j}{n}\right)$$

to inoculate and

$$c_p^0(i,\vec{a}) = L\frac{k_i}{n} + F\left(s_{p_i}^1 C + s_{p_i}^0 L\frac{k_i}{n}\right)$$

to not buy the protection.

In order that p_i inoculates it must hold that

$$\begin{aligned}
c_p^0(i,\vec{a}) &> c_p^1(i,\vec{a}) \\
L\frac{k_i}{n} + F\cdot s_{p_i}^0 L\frac{k_i}{n} &> C + F\cdot\sum_{p_j\in S_{p_i}^0} L\frac{k_j}{n} \\
L\frac{k_i}{n}(1+Fs_{p_i}^0) &> C + FL\frac{1}{n}\cdot\sum_{p_j\in S_{p_i}^0} k_j \\
k_i(1+Fs_{p_i}^0) &> Cn/L + F\cdot\sum_{p_j\in S_{p_i}^0} k_j \\
k_i &> \frac{Cn/L + F\cdot\sum_{p_j\in S_{p_i}^0} k_j}{1+Fs_{p_i}^0}.
\end{aligned}$$

□

A question of main interest is of course, under which circumstances the introduction of the social aspect to the Virus Inoculation Game leads to an improvement (or impairment, respectively) with respect to the total cost. The answer to this question is given by the following theorem.

Theorem 4.2. The value of the Windfall of Friendship is for every instance I always at least 1 and at most $PoA(I)$.

Proof of Theorem 4.2. *Lower bound.* The outline of the proof for the lower bound is as follows: We start with an instance I where $F > 0$ and let the game converge into a worst Nash equilibrium. Then we set $F = 0$ and examine the changes that occur to the just reached Nash equilibrium.

Let α be a worst Nash equilibrium in the social game. If α is also a Nash equilibrium in the same instance with $F = 0$ then we are done. Otherwise there is at least one player p_i that would decide differently about his strategy.

Assume p_i is insecure but would be secure in the original game. Then p_i's attack component has on the one hand to be at least $k_i' > Cn/L$ (cf [1]) and on the other hand at most $k_i'' = (Cn/L + F \cdot \sum_{p_j \in S^0_{p_i}} k_j)/(1 + Fs^0_{p_i}) \leq (Cn/L + Fs^0_{p_i}(k_i'' - 1))/(1 + Fs^0_{p_i}) \Leftrightarrow k_i'' \leq Cn/L - Fs^0_{p_i}$ (cf Theorem 4.1). But this is of course not possible.

It remains the case where p_i is secure but would be insecure in the original game. Note that, as every node has the same bound for the size of its attack component when $F = 0$, this switch-over cannot force any other player to change its strategy as well. Furthermore, only the costs of the players inside p_i's attack component are affected by it. The total cost of this attack component raises by at least

$$x = \underbrace{\frac{k_i}{n}L - C}_{p_i} + \underbrace{\sum_{p_j \in S^0_{p_i}} \frac{k_i}{n}L - \frac{k_j}{n}L}_{p_i\text{'s insecure neighbors}} = \frac{k_i}{n}L - C + \frac{L}{n}(s^0_{p_i}k_i - \sum_{p_j \in S^0_{p_i}} k_j).$$

As p_i remains secure in the social adaption, we know that $C + F\sum_{p_j \in S^0_{p_i}} \frac{k_j}{n}L \leq \frac{k_i}{n}L + F\sum_{p_j \in S^0_{p_i}} \frac{k_i}{n}L$ and therefore

$$\begin{aligned} x &\geq \frac{k_i}{n}L - \frac{k_i}{n}L - F\sum_{p_j \in S^0_{p_i}} \frac{k_i}{n}L + F\sum_{p_j \in S^0_{p_i}} \frac{k_j}{n}L + \frac{L}{n}s^0_{p_i}k_i - \frac{L}{n}\sum_{p_j \in S^0_{p_i}} k_j \\ &= (1-F)\frac{L}{n}s^0_{p_i}k_i - (1-F)\frac{L}{n}\sum_{p_j \in S^0_{p_i}} k_j \\ &\geq (1-F)\frac{L}{n}(s^0_{p_i}k_i - s^0_{p_i}(k_i - 1)) \\ &\geq 0. \end{aligned}$$

Clearly, after such a step we might no longer be in a friendship Nash equilibrium. However, as p_i will not change his state anymore and the initial point remains intact for all other players, our considerations still hold for further steps too. Hence, for every worst friendship Nash equilibrium there exists a Nash equilibrium which is at least as expensive.

Upper bound. As for the upper bound, it holds for any instance I that

$$WoF(I, F) = \frac{Cost(I, \vec{a}_{worstNE})}{Cost(I, \vec{a}_{worstFNE})} \leq \frac{Cost(I, \vec{a}_{worstNE})}{Cost(I, \vec{a}_{OPT})} = PoA(I)$$

□

Having this result it seems intuitive that the Windfall of Friendship is not only at least 1 but even monotonically increasing in F. But this is actually not always true as the following observation shows.

Observation 4.3. The Windfall of Friendship $WoF(I, F)$ is not for every instance I monotonically increasing in F.

As an example, consider the star graph S_n with $n = 13$, i.e. one center node and 12 boundary nodes. We set $C = 1$ and $L = 4$ and compare the two cases where $F = 0.1$ and $F = 0.9$ respectively.

Because we are interested in the worst Nash equilibrium, we have to assume that the center node is insecure. As given by Definition 3.5, the perceived cost of an insecure boundary node p_i is then

$$c_p^0(i, \vec{a}) = L\frac{n_0 + 1}{n} + FL\frac{n_0 + 1}{n} = (1 + F)(n_0 + 1)\frac{L}{n}$$

where n_0 is the number of insecure boundary nodes. If such an outer node changes its strategy, i.e. protects itself, its perceived cost gets

$$c_p^1(i, \vec{a}) = C + FL\frac{n_0}{n}.$$

Accordingly, for the center node p_j itself we have

$$c_p^0(j, \vec{a}) = L\frac{n_0 + 1}{n} + Fn_0L\frac{n_0 + 1}{n} + F(n-1-n_0)C = (1+Fn_0)(n_0+1)\frac{L}{n} + F(n-1-n_0)C$$

when it is insecure and

$$c_p^1(j, \vec{a}) = C + FL\frac{n_0}{n} + F(n - 1 - n_0)C$$

when it is protected.

If we want to let the game converge to the worst Nash equilibrium, i.e. we try to let only boundary nodes inoculate, it is easy to calculate that in the case where $F = 0.9$ all but one boundary node will inoculate. After this, neither the last outer node nor the center node will protect itself, resulting in a Nash equilibrium with total cost $(n - 2)C + 2L\frac{2}{n} = 12.23$.

However, in the case where $F = 0.1$ only 10 boundary nodes will inoculate until all boundary nodes are content with their strategies. But now the attack component is still large enough for the center node to have an advantage protecting itself, naturally causing all boundary nodes to then change back to their insecure states. In other words, here the worst Nash equilibrium is actually the best (and only) Nash equilibrium with a total cost of only $C + (n-1)\frac{L}{n} = 4.69$.

A further general result is stated by the following observation. We do not give a proof here but instead refer to the clique as an example (cf Lemma 5.2 and Lemma 5.3).

Observation 4.4. The social optimum ist not always a friendship Nash equilibrium.

Reasoning about best and worst Nash equilibria raises the question, how difficult it is to compute such Nash equilibria. We can generalize the proof given in [1] and show that computing the cheapest and the most expensive friendship Nash equilibrium is hard.

Theorem 4.5. Computing the best and the worst pure friendship Nash equilibrium is $\mathcal{NP}$-complete for any $0 \leq F \leq 1$.

Proof of Theorem 4.5. We prove this theorem by reductions from the Vertex Cover problem and the Independent Dominating Set problem. Namely, we show that answering the question whether there exists a pure friendship Nash equilibrium with cost less than k, or more than k respectively, is at least as hard as solving Vertex Cover or Independent Dominating Set. Note that verifying whether a proposed solution is correct can be done in polynomial time, hence the problems are in $\mathcal{NP}$.

Fix some graph $G = (V, E)$ and set $C = 1$ and $L = n/1.5$. We show first that the following two conditions are necessary and sufficient for friendship Nash equilibria: (a) all neighbors of an insecure node are secure, and (b) every inoculated node has at least one insecure neighbor.

As $C > L/n$ (cf Observation 3.4) condition (b) holds. To see that condition (a) holds as well, assume that there is an attack component of size at least 2. An insecure node p_i in this attack component bears the cost $\frac{k_i}{n}L + F(s^1_{p_i}C + s^0_{p_i}\frac{k_i}{n}L)$. Changing its strategy reduces its cost by at least $r_{p_i} = \frac{k_i}{n}L + Fs^0_{p_i}\frac{k_i}{n}L - C - Fs^0_{p_i}\frac{k_i-1}{n}L = \frac{k_i}{n}L + Fs^0_{p_i}\frac{1}{n}L - C$. By our assumption of $k_i \geq 2$, and hence $s^0_{p_i} \geq 1$, it holds that $r_{p_i} > 0$, resulting in p_i becoming secure. Hence, condition (a) holds as well.

We now argue that G has a vertex cover of size k if and only if the virus game has a friendship Nash equilibrium with k or fewer secure nodes, or equivalently an equilibrium with social cost at most $Ck + (n-k)L/n$, as each insecure node must be in a component of size 1 and contributes exactly L/n expected cost. Given a minimal vertex cover $V' \subseteq V$, observe that installing the software on all nodes in V' satisfies condition (a) because V' is a vertex cover and (b) because V' is minimal. Conversely, if V' is the set of secure nodes in a friendship Nash equilibrium, then V' is a vertex cover by condition (a) which is minimal by condition (b).

For the worst friendship Nash equilibrium, we consider an instance of the Independent Dominating Set problem. Given an independent dominating set $V' \subseteq V$, installing the software on all nodes except the nodes in V' satisfies condition (a) because V' is independent and (b) because V' is a dominating set. Conversely, the insecure nodes in any friendship Nash equilibrium are independent by condition (a) and dominating by condition (b). This shows that G has an independent dominating set of size at most k if and only if it has a friendship Nash equilibrium with at least $n-k$ secure nodes.

□

5 Windfall of Friendship for selected graphs

We now establish some results for the Windfall of Friendship in special graphs.

5.1 Complete Graph

As a most simple topology, we consider the complete graph (the *clique* K_n) where all players are connected to each other.

First consider the classic setting where nodes do not care about their neighbors. We have the following result:

Lemma 5.1. In the clique, there are two Nash equilibria with total cost

- $Cost(K_n, \vec{a}_{NE1}) = (n - \lceil Cn/L - 1 \rceil)C + (\lceil Cn/L - 1 \rceil)^2 L/n$, and
- $Cost(K_n, \vec{a}_{NE2}) = (n - \lfloor Cn/L \rfloor)C + (\lfloor Cn/L \rfloor)^2 L/n$.

If $\lceil Cn/L - 1 \rceil = \lfloor Cn/L \rfloor$, there is only one Nash equilibrium.

Proof of Lemma 5.1. Let $\vec{a}$ be a Nash equilibrium. Consider an inoculated node p_i and an insecure node p_j. Obviously, it holds that $c_a(i, \vec{a}) = C$ and $c_a(j, \vec{a}) = L\frac{k_j}{n}$. Note that k_j is the total number of insecure nodes in the clique. In order for p_i to remain inoculated, it must hold that $C \leq (k_j + 1)L/n$, so $k_j \geq \lceil Cn/L - 1 \rceil$; for p_j to remain insecure, it holds that $k_j L/n \leq C$, so $k_j \leq \lfloor Cn/L \rfloor$. As the total social cost in the clique is given by $Cost(K_n, \vec{a}) = (n - k_j)C + k_j^2 L/n$, the claim follows.

□

Observe that while in the Nash equilibrium, the size of the attack component is roughly twice the size of the attack component of the social optimum, as $Cost(K_n, \vec{a}) = (n - k_j)C + k_j^2 L/n$ is minimized for $k_j = \frac{1}{2}Cn/L$.

Lemma 5.2. In the social optimum, the size of the attack component is $\lfloor \frac{1}{2}Cn/L \rfloor$ or $\lceil \frac{1}{2}Cn/L \rceil$, yielding a total social cost of $Cost(K_n, \vec{a}_{OPT}) = (n - \lfloor \frac{1}{2}Cn/L \rfloor)C + (\lfloor \frac{1}{2}Cn/L \rfloor)^2 \frac{L}{n}$ or $Cost(K_n, \vec{a}_{OPT}) = (n - \lceil \frac{1}{2}Cn/L \rceil)C + (\lceil \frac{1}{2}Cn/L \rceil)^2 \frac{L}{n}$, respectively.

In order to compute the Windfall of Friendship, we have to study Nash equilibria in the social adaption first.

Lemma 5.3. In the clique, there are two friendship Nash equilibria with total cost

- $Cost(K_n, \vec{a}_{FNE1}) = (n - \lceil \frac{Cn/L-1}{1+F} \rceil)C + (\lceil \frac{Cn/L-1}{1+F} \rceil)^2 L/n$, and
- $Cost(K_n, \vec{a}_{FNE2}) = (n - \lfloor \frac{Cn/L+F}{1+F} \rfloor)C + (\lfloor \frac{Cn/L+F}{1+F} \rfloor)^2 L/n$.

If $\lceil (Cn/L - 1)/(1 + F) \rceil = \lfloor (Cn/L + F)/(1 + F) \rfloor$, there is only one Nash equilibrium.

Proof of Lemma 5.3. From Theorem 4.1 we know that in a Nash equilibrium the attack component in which a secure node p_i lies has size at least $k_i = k_j + 1 \geq (Cn/L + Fk_j^2)/(1 + Fk_j)$ where k_j is the number of insecure nodes. Some simple calculations then give that $k_j \geq \lceil (Cn/L - 1)/(1 + F) \rceil$. Similarly, for an insecure node p_j it holds that $k_j \leq (Cn/L + F(k_j - 1)^2)/(1 + F(k_j - 1))$ which yields that $k_j \leq \lfloor (Cn/L + F)/(1 + F) \rfloor$.

Given these bounds on the total number of insecure nodes in a Nash equilibrium, the social cost can be computed by replacing k_j in the total social cost formula $Cost(K_n, \vec{a}) = (n - k_j)C + k_j^2 L/n$. As the difference between the upper and the lower bound for k_j is at most 1, there are at most two Nash equilibria and the claim follows.

□

Armed with the social costs of the different equilibria, we have the following theorem.

Theorem 5.4. In the clique, the Windfall of Friendship is at most $WoF(K_n, F) = 4/3$. Moreover, there are problem instances where the worst Nash equilibrium in the social adaption is indeed a factor 4/3 better than the worst Nash equilibrium in the original game.

Proof of Theorem 5.4. *Upper Bound.* We first derive an upper bound for the Windfall of Friendship $WoF(K_n, F)$.

$$\begin{aligned} WoF(K_n, F) &= \frac{Cost(K_n, \vec{a}_{worstNE})}{Cost(K_n, \vec{a}_{worstFNE})} \\ &\leq \frac{Cost(K_n, \vec{a}_{worstNE})}{Cost(K_n, \vec{a}_{OPT})} \\ &\leq \frac{(n - \lceil Cn/L - 1 \rceil)C + (\lfloor Cn/L \rfloor)^2 \frac{L}{n}}{(n - \frac{1}{2}Cn/L)C + (\frac{1}{2}Cn/L)^2 \frac{L}{n}} \end{aligned}$$

as the optimal social cost (cf Lemma 5.2) is smaller or equal to the social cost of a friendship Nash equilibrium.

Simplifying this expression yields

$$\begin{aligned} WoF(K_n, F) &\leq \frac{n(1 - C/L)C + C^2 n/L}{n(1 - \frac{1}{2}C/L)C + \frac{1}{4}C^2 n/L} \\ &= \frac{1}{1 - \frac{1}{4}C/L}. \end{aligned}$$

This term is maximized for $L \to C$, implying that $WoF(K_n, F) \leq 4/3$.

Lower Bound. We now show that the ratio between the equilibria can really be as large as 4/3.

$$\begin{aligned} WoF(K_n, F) &= \frac{Cost(K_n, \vec{a}_{worstNE})}{Cost(K_n, \vec{a}_{worstFNE})} \\ &\geq \frac{(n - \lfloor Cn/L \rfloor)C + (\lceil Cn/L - 1 \rceil)^2 \frac{L}{n}}{(n - \lceil (Cn/L - 1)/(1+F) \rceil)C + (\lfloor (Cn/L + F)/(1+F) \rfloor)^2 \frac{L}{n}}. \end{aligned}$$

For $F \to 1$, $C = 1$ and $L \to 1$ we get

$$WoF(K_n, F) \geq (4n^2 - 8n + 4)/(3n^2 + 4n + 1).$$

For $n \geq 1$, the supremum of this function is 4/3.

□

It is not clear that there is always a pure friendship Nash equilibrium for general graphs. However, this indeed holds for complete graphs. Furthermore, if the players asynchronously change their strategies in a best response manner, then the system quickly converges to such a friendship Nash equilibrium.

Theorem 5.5. In every clique K_n, a friendship Nash equilibrium will be reached in at most $n - 1$ strategy profile changes.

Proof of Theorem 5.5. Consider a clique K_n with n_0 insecure and $n_1 = n - n_0$ secure nodes. As all players are connected to each other, an insecure player inoculates if and only if it is better to be secure in the situation with $n_0 - 1$ insecure and $n_1 + 1$ secure players than to be insecure in the initial situation. Hence, after this step, no secure player wants to be insecure.

Vice versa, a secure player gets insecure if and only if it is preferable to be insecure in the situation with $n_0 + 1$ insecure and $n_1 - 1$ secure players than to be secure in the initial situation. Hence, after a secure player switches his state, no insecure player wants to be secure.

Therefore, every player changes his strategy at most once. Moreover, as we exclude the possibility of totally (in)secure networks by Observation 3.4, the last player can never change his strategy.

□

5.2 Star Graph

Observe that in the star topology S_n the social welfare is maximized if the center node inoculates and all other nodes do not. The total inoculation cost then is C and the attack components are all of size 1, yielding a total social cost of $Cost(S_n, \vec{a}_{OPT}) = C + (n-1)L/n$.

Lemma 5.6. In the star graph S_n, the social optimum has cost $Cost(S_n, \vec{a}_{OPT}) = C + (n-1)L/n$.

As we will see, the situation where only the center node is inoculated also constitutes a Nash equilibrium. However, there are more Nash equilibria.

Lemma 5.7. In the star graph, there are three Nash equilibria with total cost

- $Cost(S_n, \vec{a}_{NE1}) = C + (n-1)L/n$,
- $Cost(S_n, \vec{a}_{NE2}) = (n - \lceil Cn/L - 1 \rceil)C + (\lceil Cn/L - 1 \rceil)^2 L/n$, and
- $Cost(S_n, \vec{a}_{NE3}) = (n - \lfloor Cn/L \rfloor)C + (\lfloor Cn/L \rfloor)^2 L/n$.

If $\lceil Cn/L - 1 \rceil = \lfloor Cn/L \rfloor$, there are only two Nash equilibria.

Proof of Lemma 5.7. Clearly, the center node being the only secure node is a Nash equilibrium: If the center node p_i changes its strategy, its cost changes from C to L; if a boundary node becomes secure, its cost changes from L/n to C. But by Observation 3.4 these changes are unprofitable. The social cost of this Nash equilibrium is $Cost(S_n, \vec{a}_{NE1}) = C + (n-1)L/n$.

For the other Nash equilibria we have to assume that the center node is not inoculated. Let the number of insecure boundary nodes be n_0. In order for a secure node to remain secure, it must hold that $C \leq (n_0 + 2)L/n$, so $n_0 \geq \lceil Cn/L - 2 \rceil$. In order for an insecure node to remain insecure, it must hold that $(1 + n_0)L/n \leq C$, so $n_0 \leq \lfloor Cn/L - 1 \rfloor$. Therefore, we can conclude that there are at most two other Nash equilibria, one with $\lceil Cn/L - 1 \rceil$ and one with $\lfloor Cn/L \rfloor$ many insecure nodes. Note that these Nash equilibria are distinct, i.e. the clique has three NE, if and only if Cn/L is an integer value. The total social cost follows by substituting n_0 in the total social cost function.

□

We now turn our attention again to players who care about their neighbors.

Lemma 5.8. In the star graph, there are at most three friendship Nash equilibria with total cost

- $Cost(S_n, \vec{a}_{FNE1}) = C + (n-1)L/n$,
- $Cost(S_n, \vec{a}_{FNE2}) = (n - \lceil Cn/L - 1 - F \rceil)C + (\lceil Cn/L - 1 - F \rceil)^2 L/n$, and
- $Cost(S_n, \vec{a}_{FNE3}) = (n - \lfloor Cn/L - F \rfloor)C + (\lfloor Cn/L - F \rfloor)^2 L/n$.

If $\lceil Cn/L - 1 - F \rceil = \lfloor Cn/L - F \rfloor$, there are only at most two Nash equilibria.

Proof of Lemma 5.8. First, observe that having an inoculated center node and all other nodes insecure constitutes a friendship Nash equilibrium. In this case, in order for the center node to remain inoculated, it must hold that $C+F(n-1)L\frac{1}{n} \leq nL/n + F(n-1)L\frac{n}{n} = L + F(n-1)L$. All boundary nodes remain insecure as long as $L/n + FC \leq C + FC \Leftrightarrow L/n \leq C$. Both conditions are always true as stated by Observation 3.4. We have $Cost(S_n, \vec{a}_{FNE1}) = C + (n-1)L/n$.

If the center node is not inoculated, we have n_0 insecure and $n - n_0 - 1$ inoculated boundary nodes. In order for a secure boundary node to remain secure, it must hold that $C + F\frac{n_0+1}{n}L \leq \frac{n_0+2}{n}L + F\frac{n_0+2}{n}L$, so $n_0 \geq \lceil Cn/L - 2 - F \rceil$. For an insecure boundary node, it must hold that $\frac{n_0+1}{n}L + F\frac{n_0+1}{n}L \leq C + F\frac{n_0}{n}L$, so $n_0 \leq \lfloor Cn/L - 1 - F \rfloor$. Inserting these two values into the social cost function yields the claim.

□

To explain the term 'at most' in Lemma 5.8, consider the following: The center node is insecure and every boundary node is content with its strategy. But because of its exposed position, the center node still wants to become secure, naturally causing the system to converge into *FNE1*, the best friendship Nash equilibrium. In other words, there are instances for which *FNE1* is the only friendship Nash equilibrium. We already made use of this phenomenon in Section 4 to show that $WoF(I, F)$ is not always monotonically increasing in F.

Knowning about this phenomenon, we would like to know of course under which circumstances it occurs. The next lemma gives us the necessary and sufficient condition.

Lemma 5.9. Let $F > 0$. The instance S_n has only one (the best) friendship Nash equilibrium if and only if

$$\lfloor Cn/L - 1 - F \rfloor - \lfloor \frac{1}{2F}(\sqrt{1 - 4F(1 - Cn/L)} - 1) + 1 \rfloor \geq 0$$

Proof of Lemma 5.9. As discussed above, S_n has only one Nash equilibrium if every (insecure) boundary node is content with its chosen strategy but the insecure center node would rather inoculate. In order for an insecure boundary node to remain insecure we must have (cf Proof of Lemma 5.8) that $n_0 \leq \lfloor Cn/L - 1 - F \rfloor$ (Condition I) and the insecure center node wants to inoculate if and only if $C + F(n - n_0 - 1)C + Fn_0 \frac{1}{n} L < (n_0 + 1)\frac{L}{n} + F(n - n_0 - 1)C + Fn_0 \frac{n_0+1}{n} L \Leftrightarrow Fn_0^2 + n_0 + 1 - Cn/L > 0 \Leftrightarrow n_0 \geq \lfloor \frac{1}{2F}(\sqrt{1 - 4F(1 - Cn/L)} - 1) + 1 \rfloor$ (Condition II).

Note that it is not important whether the condition for a secure boundary node to remain secure, i.e. $n_0 \geq \lceil Cn/L - 2 - F \rceil$ (Condition III), holds or not. If it does, then the center node is the only player that wants to change his strategy. If it does not, then maybe a secure boundary node p_i gets insecure first. As this increases n_0 by 1, Condition II still holds. Moreover, we thus have $n_0 < \lceil Cn/L - 2 - F \rceil \leq \lfloor Cn/L - 1 - F \rfloor$ before p_i gets insecure and so $n_0' = n_0 + 1 < \lfloor Cn/L - F \rfloor \Leftrightarrow n_0' \leq \lfloor Cn/L - 1 - F \rfloor$ afterwards. Hence, Condition I still holds as well.

Therefore there is only one Nash equilibrium if and only if there exists an integer n_0 such that $\lfloor \frac{1}{2F}(\sqrt{1 - 4F(1 - Cn/L)} - 1) + 1 \rfloor \leq n_0 \leq \lfloor Cn/L - 1 - F \rfloor$.

□

Note that it is indeed possible for a star graph to have exactly three friendship Nash equilibria. As an example, consider the star graph S_n with $n = 9$ nodes and let $C = 1$, $L = 4$ and $F = 0.25$. Then, as $Cn/L - F$ is an integer value, Lemma 5.8 gives us that all three friendship Nash equilibria are distinct. It remains to check that the system does not necessarily converge to the best FNE. As $\lfloor Cn/L - 1 - F \rfloor - \lfloor \frac{1}{2F}(\sqrt{1 - 4F(1 - Cn/L)} - 1) + 1 \rfloor = 1 - \lfloor 2(\sqrt{9/4} - 1) + 1 \rfloor = -1 < 0$ and so the condition from Lemma 5.9 does not hold, there really exist all three FNE in S_n.

Now we are ready to give some upper bounds on the Windfall of Friendship for star graphs. We have the following result:

Theorem 5.10. If the condition from Lemma 5.9 holds, i.e. there is only one friendship Nash equilibrium, then the Windfall of Friendship in the star graph S_n

is equal to the Price of Anarchy $PoA(S_n) < (n+1)C/L$. Otherwise the Windfall of Friendship is less than $(n+1)/(n-3)$.

Proof of Theorem 5.10. If the condition from Lemma 5.9 holds, we have

$$\begin{aligned}
WoF(S_n, F) &= \frac{Cost(S_n, \vec{a}_{worstNE})}{Cost(S_n, \vec{a}_{worstFNE})} \\
&= \frac{Cost(S_n, \vec{a}_{worstNE})}{Cost(S_n, \vec{a}_{OPT})} = PoA(S_n) \\
&\leq \frac{(n - \lceil Cn/L - 1 \rceil)C + (\lfloor Cn/L \rfloor)^2 L/n}{C + (n-1)L/n} \\
&\leq \frac{(n+1)C}{C + (n-1)L/n} \\
&< \frac{(n+1)C}{L}.
\end{aligned}$$

Otherwise, i.e. if the game does not necessarily converge to the best friendship Nash equilibrium, we have

$$\begin{aligned}
WoF(S_n, F) &= \frac{Cost(S_n, \vec{a}_{worstNE})}{Cost(S_n, \vec{a}_{worstFNE})} \\
&\leq \frac{(n - \lceil Cn/L - 1 \rceil)C + (\lfloor Cn/L \rfloor)^2 L/n}{(n - \lfloor Cn/L - F \rfloor)C + (\lceil Cn/L - 1 - F \rceil)^2 L/n} \\
&\leq \frac{(n+1)C}{nC + FC - 2C(1+F) + (1+F)^2 L/n} \\
&< \frac{(n+1)C}{C(n + F - 2(1+F))} \\
&< \frac{n+1}{n-3}.
\end{aligned}$$

□

Similar to Theorem 5.5, we also have a fast convergence to a friendship Nash equilibrium in star graphs.

Theorem 5.11. In every star graph S_n, a friendship Nash equilibrium will be reached in at most $2n-3$ strategy profile changes. This bound is tight.

Proof of Theorem 5.11. To prove this theorem, we first state the following four observations.

Claim 5.12. During the whole process, it holds that

1. as soon as the center node is secure, no boundary node will inoculate anymore,
2. if the center node eventually inoculates, it will never change back to be insecure again,
3. if the center node switches its state from secure to insecure, no boundary node will inoculate anymore,
4. if a boundary node switches its state from secure (insecure) to insecure (secure), no boundary node will protect (unprotect) itself anymore, given that the center node does not switch its state.

Proof of Claim 5.12. 1. For the first observation we have that if the center node is secure, then every boundary node is isolated and so, by Observation 3.4, no one of them wants to be secure.

2. The center node prefers to be secure if and only if the number of insecure boundary nodes reaches some given limit. After the center node inoculates, it holds by the first observation that this number can only increase. Hence, the limit is reached further on and the center node therefore stays secure, which proves the second observation.

3. Let p_i be the secure center node and p_j one of the n_0 insecure boundary nodes. p_i gets insecure if and only if $C + F(\frac{n_0}{n}L) > \frac{n_0+1}{n}L + F(n_0\frac{n_0+1}{n}L) \Leftrightarrow Cn/L > n_0 + 1 + F{n_0}^2$. In order for p_j to then change its state it must hold that $\frac{n_0+1}{n}L + F\frac{n_0+1}{n}L > C + F\frac{n_0}{n}L \Leftrightarrow Cn/L < n_0 + 1 + F$. As this is not possible, the third observation holds as well.

4. Finally, a boundary node switches its state only if it is preferable to do so. But another boundary node which afterwards redecides in the other direction contradicts this condition. Thus, the last observation holds as well.

□

Let n_1 be the number of secure boundary nodes. The last observation states that we can either increase or decrease n_1 until a strategy change of the center node occurs, whereas by the first and the third observation n_1 can only be decreased after such a strategy change. Actually, the first observation even says that if the center node is already inoculated in the beginning, then n_1 can only be decreased during the whole process.

Note that n_1 is always a positive integer of at most $n-1$. However, if n_1 is increased until its value is indeed $n-1$, then the center node is isolated and we have reached a Nash equilibrium.

Thus it is clear that there is no longer path to a Nash equilibrium than starting with the center node being insecure, increasing n_1 as long as possible, switching the strategy of the center node, and then decreasing n_1 again. Note that by the second observation it is not possible that the center node changes its strategy again during the decreasing phase.

Therefore, a friendship Nash equilibrium is reached after at most $(n-2)+1+(n-2) = 2n-3$ strategy profile changes.

□

6 Conclusion

We have extended the Virus Inoculation Game by adding the aspect of friendship: Players are no longer totally selfish but they also care to some degree about their neighbors. We have seen that introducing friendship makes players act more conservatively, i.e. they have an increased willingness to inoculate. We could prove that this cautiousness never impairs the whole community. However, even if it is quite a bit surprising, the improvement to the total social cost is not always monotonically increasing in the degree of how much the players care for their friends. For a somehow similar phenomenon in the real world, consider a network of people all of which decide to bake some Christmas cookies for their friends. This is nice and the community gets happier, but if everybody gives away a really large amount of self-made cookies, then the community gets severely surfeited.

The introduced social aspect was restricted by choosing the radius of friendship to be very small, i.e. friendship only exists between nodes that are directly connected, which allows for extension in further examination. For example, one could consider friendship over multiple hops or let the metric distance between two players decide about their degree of friendship, i.e. porting the Virus Inoculation Game into the Euclidean space.

The game itself can also be extended of course. For example, a virus might not infect unlimitedly many insecure players that are adjacent to an already infected player. Or there might be different kinds of viruses and different kinds of nodes, which are resistent to some viruses but not to others. Furthermore, one could also modify the underlying model by introducing Byzantine players in this social adaption as well.

Apart from all these propositions and more closely continuing the work done in this semester thesis, one could try to establish a proof for convergence to a friendship Nash equilibrium in general graphs. This question is substantially more involved than for the original game because in the social adaption the perceived cost of a player depends on his neighbors and their attack components. Hence, every player has an own maximal allowed size of his attack component and so a player may become insecure even though another player is thereby forced to inoculate, which is not possible in the totally selfish game.

Moreover, we have calculated the Windfall of Friendship for two special topologies, but not for other or even general graphs. The difficulty herein is on the one hand the task of finding the worst Nash equilibrium at all, and on the other hand that the players need to have a relatively good knowledge about the topology of the network to be able to compute their perceived cost. Hence, to analyze such a network, we also have to have a good knowledge about its topology.

Last but not least, introducing friendship is not only an interesting aspect for the Virus Inoculation Game but can lead to various new insights and improvements in other games, and even in diverse non-computational fields, as well.

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