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Semester Thesis

Byzantine Caching Game

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Abstract

We analyze the impact of Byzantine players in a game modeling the decentralized caching of resources. In [1] such a game was introduced as Selfish Caching Game. We extend this game by Byzantine players and examine its efficiency by analyzing its Price of Anarchy and its Price of Malice. For the line and grid topology we provide tight bounds on these ratios. In our model the selfish server nodes incur either cost for replicating resources or cost for accessing a remote replica.

1 Introduction

Peer-to-peer systems offer the possibility to distribute computing power, bandwidth or memory on its independent peers. While distributed fileshare systems are already widely used, also web caches and peer-to-peer caches have become popular during the last few years. New promising systems like Kangoo provide a global, distributed storage system.

Distributed storage systems can be modeled by a game theoretic approach, where the players are connected among each other according an underlying topology. Chun et al. [1] introduced such a game as the Selfish Caching Game. The selfish behavior of the players can degrade the performance of such a system strongly. However, selfishness is not the only problem, malicious Byzantine adversaries can have an extensive impact on the performance as well.

In this thesis we examine the effects of malicious Byzantine players on the efficiency of the Selfish Caching Game with different underlying topologies. In our model the Byzantine players seek to lower the utility of the entire system, independently of their own cost. The Price of Malice gives an indication on the effect of Byzantine players, whereas the Price of Anarchy shows the influence of selfish behavior on socially optimal performance of a system. These two measurements have different reference points - the Price of Anarchy relates the social welfare generated by players acting in an egoistic manner to an optimal solution obtained by perfectly collaborating participants, whereas the Price of Malice's reference point is the welfare achieved by an entirely selfish system. We analyze these two variables for the underlying topologies star, line and grid. Furthermore we perform first examinations in the d-dimensional grid.

2 Related Work

We took over the Caching Game from [1]. Their examinations quantifying the cost of lack of coordination when servers behave selfishly were of interest for us. They give results on the Price of Anarchy for the star and line topology. We extend the Caching Game with malicious Byzantine players and analyze their impact on the overall performance.
Additional studies of the inherent loss of efficiency in a system caused by the participant's selfishness in networks were performed by Schmid et al. [2]. They investigated the impact of selfish neighbor selection on the quality of the resulting network topologies.
Malicious Byzantine players were introduced by Moscibroda et al. [3]. They have performed analysis in a Virus Inoculation Game. For their exami-

nations they quantified how much the presence of Byzantine players can deteriorate or even improve the social welfare of the distributed system by analyzing and comparing its Price of Malice and Price of Anarchy.

3 Model

3.1 Byzantine Caching Game

We obtained the basic model of the Selfish Caching Game from [1] and extended it with Byzantine players. We abstracted the caching problem as follows. There is one object which is cached at several players $i \in N$. S denotes the set of selfish and B the set of Byzantine players. It holds $|S| + |B| = |N|$. The distance between servers, or rather players, can be represented as a distance matrix D (i.e., d_{ij} is the distance from player i to player j) which models the underlying network topology. Each player has two choices, he can cache the object or access a remote copy. The choices of the players can be summarized by a strategy profile $\vec{a} \in \{0,1\}^n$, where $a_i = 1$ signifies that player i caches the object, and $a_i = 0$ that it does access a copy. Therefore, the cost of a selfish player i is

$$cost_i(\vec{a}) = \underbrace{a_i \cdot \alpha}_{caching\ cost} + \underbrace{(1 - a_i) \cdot d_{ij}.}_{accessing\ cost}$$

The *social cost* of a strategy profile $\vec{a}$ is the sum of all individual costs, $Cost(\vec{a}) = \sum_{i \in S} cost_i \vec{a}$, where S denotes the set of selfish players. To further formulate this equation we define the *Influence Area* I_a of a caching node a. I_a contains all nodes that access the cached copy at node a. We sum up the costs occurring in all influence areas to get the total social cost

$$Cost = \underbrace{\alpha \cdot |I|}_{CachingCost} + \underbrace{\sum_I (\sum_{i \in I_a} d_{i,a}).}_{AccessingCost} \tag{1}$$

3.2 Byzantine Game Theory

The *Price of Anarchy* designates the ratio of the social cost of the worst Nash equilibrium to the social optimum, $PoA = \frac{Cost_{NE}}{Cost_{OPT}}$. As we consider also Byzantine players we have to comprise their effect in the analysis. From [3] we obtain the following definitions for the Byzantine game theory. In a *Byzantine Nash equilibrium* no selfish player has an incentive to change his strategy if the strategies of all other (selfish and Byzantine) players are fixed. The *Price of Byzantine Anarchy* is the ratio between the worst-case social cost of a Byzantine Nash equilibrium divided by the optimal social cost, $PoB(b) = \max \frac{Cost_{BNE}(b)}{Cost_{OPT}}$. To capture the ratio between the worst Byzantine Nash Equilibrium with b malicious players and the Nash Equilibrium in a purely selfish system we calculate the *Price of Malice*, $PoM(b) = \frac{PoB(b)}{PoA}$. For the latter we assume that in the *Oblivious Model* selfish players are not aware of the existence of Byzantine players. That is, they expect all other players in the system to be selfish as well and therefore do not adapt their strategy to Byzantine players. In the *Non-Oblivious Model* we assume that the selfish players act according a risk-averse strategy. The number of Byzantine players is known, but not their location or strategy. Moreover, they presume that the Byzantine players are located in worst case situation concerning their individual cost. We designate in the following with $Cost_{OPT}$ the optimal social cost, with $Cost_{NE}$ the worst case cost of a Nash equilibrium and finally with $Cost_{BNE}$ the worst case cost of a Byzantine Nash equilibrium.

4 Analysis

To analyze the Price of Anarchy and the Price of Malice for different topologies we have to provide tight bounds on the Social Optimum, the Nash equilibrium and the Byzantine Nash equilibrium. In the following sections we begin with proofing several lemmas establishing the needed bounds without and with Byzantine players. Furthermore we proceed analyzing the Byzantine behavior in the non-oblivious and the oblivious case. Finally we analyze the Price of Anarchy and the Price of Malice on behalf of the obtained findings. In the Appendix a detailed overview of the results is given.

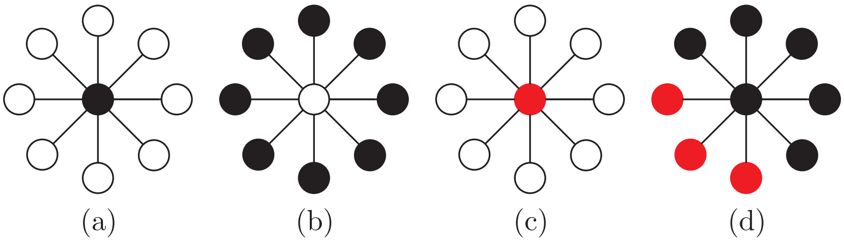


Figure 1: Star topology for $1 < \alpha < 2$. (a) Social optimum. (b) Worst case Nash equilibrium. (c) Worst case Byzantine Nash equilibrium, Oblivious Model. (d) Worst case Byzantine Nash equilibrium, Non-Oblivious Model.

4.1 Star

In the star topology the Price of Anarchy is smaller equal 2 for $\alpha > 1$. That means a central planning can be twice as good as if every player is acting selfishly. In the oblivious model the star is obviously very vulnerable to malicious players. Byzantine players easily can cause damage by caching objects at the central nodes. We can see this by the Byzantine Price of Anarchy which is infinite for $1 < \alpha < 2$ and $b \geq 1$. This leads to an infinite Price of Malice, thus the Byzantine players cause maximal damage. In the non-oblivious model the selfish nodes know about the existence of Byzantine players and therefore are able to adapt their strategy. However, they do not know about their exact location and therefore can not be sure to access a valid copy. Thus they prefer to cache the object and the Price of Malice becomes $PoM < 1$. That could indicate that the social cost are lowered by the existence of Byzantine player. Though, in the star topology this effect is mainly achieved because only the selfish players contribute to the social cost. If we compare Figure 1(b) and 1(d) we see that the two strategies are almost the same. The lower social cost is caused mainly because of the Byzantine players which do not contribute to it.

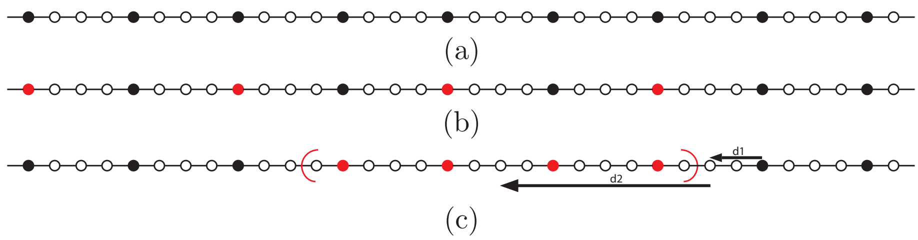


Figure 2: Line topology for $\alpha = 2$. (a) Worst case Nash equilibrium. (b) Byzantine Nash equilibrium for $b = 4$. (c) Worst case Byzantine Nash equilibrium for $b = 4$. $d1 = \alpha, d2 = \frac{b \cdot \alpha}{2}$.

4.2 Line

In the following we establish tight results for the Price of Malice of the line. For the calculations of the Social Optimum we assume the cached objects to be located at equal distances in the optimal case. Our gained results encourage this presumption, furthermore we give an outline for a proof of the optimality of a regular arrangement in the grid in Lemma 4.5.

4.2.1 Social Optimum

Lemma 4.1 (Distances in Social Optimum). *Under the assumption that the objects are cached at equal distances, it holds for the distance between two neighboring caching nodes* $\sqrt{2\alpha} \leq d < 2\sqrt{2\alpha}$ *in the Social Optimum.*

Proof. Consider a distribution of cached objects on the line at an arbitrary distance d. Starting from this general setting we want to decrease the social cost, by placing or removing objects. If d is too big we can place an additional object in the middle of the interval at node a. The social cost is then reduced in one interval by

$$\begin{aligned}
CostReductionP(d) &= AccessCost_a - CachingCost_a \\
&+2\sum(AccessCost_{old} - AccessCost_{new}) \\
&= \frac{d}{2} - \alpha + 2\sum_{i=1}^{d/4}\left(\left(\frac{d}{2} - i\right) - i\right) \\
&= \frac{d}{2} - \alpha + 2\sum\frac{d}{2} - 2\sum 2i \\
&= \frac{d}{2} - \alpha + \frac{d^2}{4} - \left(\frac{d}{2}\left(\frac{d}{4}+1\right)\right) \\
&= \frac{d^2}{8} - \alpha
\end{aligned}$$

We only place the object if $CostReductionP > 0$. Thus, only if the size of the interval is $d > 2\sqrt{2\alpha}$. If we proceed placing additional objects in the middle till $CostReductionP \leq 0$ then it holds for the intervals on the line

$$d \leq 2\sqrt{2\alpha} \tag{2}$$

If the intervals initially are too small then we remove every second object. For each removed node a the social cost is then reduced by

$$\begin{aligned} CostReductionR(d) &= CachingCost_a - AccessCost_a \\ &-2\sum(AccessCost_{new} - AccessCost_{old}) \\ &= \alpha - d - 2\sum_{i=1}^{d/2}((d-i)-i) \\ &= \alpha - \frac{d^2}{2} \end{aligned}$$

Again, the reduction of social cost has to be positive, $CostReductionR > 0$. It follows that we can lower the social cost if $d < \sqrt{2\alpha}$. If we proceed removing objects till finally $CostReductionP \leq 0$ then it holds for the intervals on the line

$$d \geq \sqrt{2\alpha} \tag{3}$$

By removing or adding objects till there is no gain we construct finally a social optimum. From (2) and (3) it follows for the size of the intervals in the social optimum $\sqrt{2\alpha} \leq d \leq 2\sqrt{2\alpha}$. □

Lemma 4.2 (Social Optimum). *The social cost of the Social Optimum on a line for* $1 < \alpha < n$ *is*

$$Cost_{OPT} \in \Theta(\sqrt{\alpha} \cdot n).$$

PROOF. We proof the upper and lower bound in turn.

Lower Bound. From (1) we obtain for the social cost of a Social Optimum

$$\begin{aligned}
Cost_{OPT}(d) &= CachingCost + AccessingCost \\
&= \alpha \cdot |I| + |I| \cdot 2 \cdot \sum_{i=1}^{d/2} i \qquad (4)\\
&= \alpha \cdot \frac{n-1}{d} + \frac{n-1}{d} \cdot 2 \cdot \frac{\frac{d}{2}(\frac{d}{2}+1)}{2} \\
&= \alpha \cdot \frac{n-1}{d} + \frac{n-1}{d} \cdot \frac{d}{2}(\frac{d}{2}+1) \qquad (5)\\
&= \alpha \cdot \frac{n-1}{d} + \frac{n-1}{2} \cdot \left(\frac{d}{2}+1\right) \qquad (6)
\end{aligned}$$

According to Lemma 4.1 it holds $d \in [d_{min}, d_{max}]$. If we look closer to (6) we detect that the *CachingCost* decreases with increasing d and *AccessingCost* decreases with decreasing d. Therefore it holds for the total social cost of the Social Optimum

$$\begin{aligned}
Cost_{OPT}(d) &\geq CachingCost(d_{max}) + AccessingCost(d_{min}) \\
&\geq \alpha \cdot \frac{n-1}{d_{max}} + \frac{n-1}{2} \cdot \left(\frac{d_{min}}{2}+1\right) \\
&\geq \alpha \cdot \frac{n-1}{\sqrt{2\alpha}} + \frac{n-1}{2} \cdot \left(\frac{2\sqrt{\alpha}}{2}+1\right) \\
&\geq \frac{\sqrt{2\alpha}}{2} \cdot (n-1) + (\sqrt{2\alpha}+1) \cdot (n-1) \\
&\in \Omega(\sqrt{\alpha} \cdot n)
\end{aligned}$$

□

Upper Bound. We obtain a optimal example for the social cost of the *Social Optimum* if we set the *Caching Cost* to the *Accessing Cost* in (5)

$$\begin{aligned}
\alpha \cdot \tfrac{n-1}{d} &= \frac{n-1}{d} \cdot \frac{d}{2}\left(\frac{d}{2}+1\right) \\
\alpha &= \frac{d}{2}\left(\frac{d}{2}+1\right) \\
0 &= \frac{d^2}{4} + \frac{d}{2} - \alpha
\end{aligned}$$

The positive solution is $d = -1 + \sqrt{1 + 4\alpha}$. Inserting this value in (4) leads us to the social cost $Cost_{OPT} = \frac{2\cdot\alpha}{-1+\sqrt{1+4\alpha}} \cdot (n-1)$ and establishes the upper bound $Cost_{OPT} \in O(\sqrt{\alpha} \cdot n)$.

□

4.2.2 Oblivious Model, Byzantine Nash Equilibrium

Lemma 4.3 (Byzantine Nash Equilibrium). *The social cost of the worst Byzantine Nash Equilibrium in the oblivious model for* $1 < \alpha < n, 0 \leq b < \frac{n}{2\alpha}$ *is*

$$Cost_{BNE} \in \Theta(\alpha \cdot s + \alpha^2 \cdot b^2).$$

PROOF. We proof the upper and lower bound in turn.

Lower Bound. We consider an example for a Byzantine Nash equilibrium which causes high social cost. The objects are cached regularly every 2α. A Byzantine node b defines the influence area I_b, which contains all nodes that would normally access the copy at b. The Byzantine players are situated successively at distances 2α, so that they form one big Byzantine influence area $I_{Bmax} = |B| \cdot I_b$ as shown in Figure 2(c). The selfish nodes lying in I_{Bmax} have to access the nearest copy outside the interval. Consequently, the Byzantine nodes cause the additional cost $AddCost_{Bmax}$. In the following calculations of $AddCost_{Bmax}$ we have to subtract the access cost of the Byzantine nodes, because they do not contribute to the social cost.

$$\begin{aligned}
AddCost_{Bmax} &= \sum_{i \in I_{Bmax}} AccessCost_i - \sum_{b_i \in I_{Bmax}} AccessCost_{b_i} \\
&= 2 \cdot \sum_{i=1}^{b \cdot 2\alpha/2} \big(\alpha + i\big) - 2 \cdot \sum_{i=1}^{b/2} \big(2\alpha + i \cdot 2\alpha\big) \\
&= 2\alpha^2 \cdot b + \alpha^2 b^2 + \alpha \cdot b - 2\alpha \cdot b - \frac{\alpha \cdot b^2}{2} - \alpha \cdot b \\
&= \alpha^2 b^2 + 2\alpha^2 \cdot b - \frac{\alpha \cdot b^2}{2} - 2\alpha \cdot b \qquad (7) \\
&\in \Omega(\alpha^2 b^2)
\end{aligned}$$

Therefore this example of a Byzantine Nash equilibrium causes the total social cost

$$\begin{aligned}
Cost_{BNE} &= CachingCost + AccessingCost + AddCost_{Bmax} \\
&= \alpha \cdot |I_S| + |I_S| \cdot \sum_{i \in I_s} d_{i,s} + AddCost_{Bmax} \\
&= \alpha \cdot \frac{s-1}{2\alpha} + 2\frac{\alpha(\alpha+1)}{2}\frac{s-1}{2\alpha} + \Omega(\alpha^2 b^2)
\end{aligned}$$

Thus we obtain for the lower bound $Cost_{BNE} \in \Omega(s \cdot \alpha + \alpha^2 b^2)$.

$\square$

Upper Bound. We designate the area which is influenced by k_i Byzantine players as a Byzantine interval $I_{B_{k_i}}$. These Byzantine players cause the cost $AddCost_{B_{k_i}}$. Let $z \in Z$ be a possible distributions of Byzantine Players among all caching players. The total caused social cost in all Byzantine intervals is $AddCost_B(z) = \sum_i AddCost_{B_{k_i}}$ with the condition $\sum_k k_i = b$.

According to (7) we obtain for the caused cost in a Byzantine interval at least $AddCost_{B_{k_i}} = \alpha^2 k_i^2 + 2\alpha^2 \cdot k_i - \frac{\alpha \cdot k_i^2}{2} - 2\alpha \cdot k_i$. Therefore the total caused cost of all Byzantine intervals is

$$AddCost_B = \sum_i AddCost_{B_{k_i}} = \alpha^2 \sum k_i^2 + 2\alpha^2 \cdot \sum k_i - \frac{\alpha}{2} \sum k_i^2 - 2\alpha \sum k_i \tag{8}$$

The first summand is dominant and needs further analysis. We show that $AddCost_B$ is maximized if all Byzantine players compose one big Byzantine interval I_{Bmax} (see Figure 2(c)), that is $\max_{z \in Z}\{AddCost_B(z)\} = AddCost_{Bmax}$.

Proof by Contradiction. We assume that $\exists z \in Z, \sum_i AddCost_{B_{k_i}}(z) > AddCost_{Bmax}$. Then we obtain from (7) and (8)

$$\sum_i^m k_i^2 > b^2$$

Inserting our constraint $\sum_i^m k_i = b$ results in

$$\begin{aligned} \textstyle\sum_i^m k_i^2 \quad & > \left(\sum_i^m k_i\right)^2 \\ \tfrac{m(m+1)(2m+1)}{6} \quad & > \left(\frac{m(m+1)}{2}\right)^2 \\ \tfrac{1}{6}(2m^3 + 3m^2 + m) \quad & > \frac{1}{4}(m^4 + 2m^3 + m^2) \end{aligned}$$

That is a contradiction, thus our assumption is wrong. The caused cost of an arbitrary Byzantine Nash equilibrium can not exceed the caused cost of the Byzantine Nash equilibrium where the Byzantine player compose one big interval, $\max_{z \in Z}\{AddCost_B(z)\} = AddCost_{Bmax}$. Therefore the upper bound equals our result of the lower bound, $Cost_{BNE} \in O(s \cdot \alpha + \alpha^2 b^2)$. $\square$

4.2.3 Non-Oblivious Model, Byzantine Nash Equilibrium

Lemma 4.4 (Byzantine Nash Equilibrium)**.** *The social cost of the Byzantine Nash Equilibrium in the non-oblivious model for* $1 < \alpha < n, 0 \leq b < \frac{n}{2\alpha}$ *is*

$$Cost_{BNE} \in \Omega\big(b \cdot s + \frac{\alpha \cdot s}{b} + \alpha^2\big).$$

Lower Bound. We give an example for a bad Byzantine Nash equilibrium in the non-oblivious model causing high social cost. Each player is risk-averse and supposes all b Byzantine players to be located in his neighborhood. By caching the objects at distances $d = 4\alpha/b$, the individual player can be sure to meet a valid copy at least after distance 2α. The Byzantine players form again one big interval I_{Bmax} and cause the additional cost

$$\begin{aligned} AddCost_{BNE} &= \sum_{i \in I_{Bmax}} AccessCost_i - \sum_{b_i \in I_{Bmax}} AccessCost_{b_i} \\ &= 2 \cdot \sum_{i=1}^{b/2 \cdot 4\alpha/b} \big(\alpha + i\big) - 2 \cdot \sum_{i=1}^{b/2} \left(\alpha + i \cdot \frac{4\alpha}{b} \cdot \frac{1}{2}\right) \\ &= 2 \cdot \sum_{i=1}^{2\alpha} \big(\alpha + i\big) - 2 \cdot \sum_{i=1}^{b/2} \left(\alpha + i \cdot \frac{2\alpha}{b}\right) \\ &= 8\alpha^2 + \alpha - \frac{3}{2}\alpha \cdot b \end{aligned}$$

Because we set the distance to $4\alpha/b$ we assume $b < 2\alpha$. Therefore we can neglect the influence of the last term and get for the additional caused cost

$$AddCost_{BNE} \in \Omega(\alpha^2)$$

The exact value of the number of selfish intervals is $|I_S| = |I| - |I_B| = |I| - b(2\alpha - 1)$. We neglect the impact of the number of Byzantine intervals and assume in our calculations for the social cost $b < s$. Thus we can set $|I_S| = |I| - |I_B| = |I|$. The social cost of this Byzantine Nash equilibrium is

$$\begin{aligned} Cost_{BNE} &= CachingCost + AccessingCost + AddCost_{BNE} \\ &= \alpha \cdot |I_S| + |I_S| \cdot \sum_{i \in I_s} d_{i,s} + AddCost_{BNE} \\ &= \alpha \cdot \frac{s-1}{\frac{4\alpha}{b}} + 2 \cdot \frac{\frac{4\alpha}{2b}\left(\frac{4\alpha}{2b} + 1\right)}{2} \cdot \frac{s-1}{\frac{4\alpha}{b}} + \Omega(\alpha^2) \\ &= \frac{1}{4}b \cdot (s-1) + \frac{\alpha}{b} \cdot (s-1) + \frac{1}{2}(s-1) + \Omega(\alpha^2) \end{aligned}$$

Thus we obtain for the lower bound in the non-oblivious model $Cost_{BNE} \in \Omega(b \cdot s + \frac{\alpha \cdot s}{b} + \alpha^2)$.

□

The following discusses a selection of interesting results, furthermore the Appendix contains an overview of all findings. We took over the bounds for the Price of Anarchy from [1]. It holds $PoA \in \Theta(\sqrt{\alpha})$ for $1 < \alpha < n$. In the oblivious model for $1 < \alpha < n, 1 \leq b < \frac{n}{2\alpha}$ we obtain for the Price of Malice $PoM \in \Theta(1 + \frac{\alpha b^2}{n})$. It is interesting to compare this with the non-oblivious case where the Price of Malice is $PoM = \Omega(\frac{b}{\alpha} + \frac{1}{b} + \frac{\alpha}{n})$. Here, it is possible to render the $PoM < 1$ under certain conditions. If $b > 2$, $b < \alpha/2$ and $n \gg \alpha$ then the Byzantine players can help to lower the total social cost compared to a purely selfish system. For $\alpha = 20, b = 5, n > 1000$ we obtain a smaller social cost for the Byzantine Nash equilibrium $Cost_{BNE} = 5.75 \cdot n + 3080$ than for the Nash equilibrium $Cost_{NE} = 10.5 \cdot n$. In the non-oblivious model the selfish players act risk-aversely, consequently the caching intervals shrink and the social cost decreases. This effect is diminished by the additional cost caused by the Byzantine players, however only for small n.

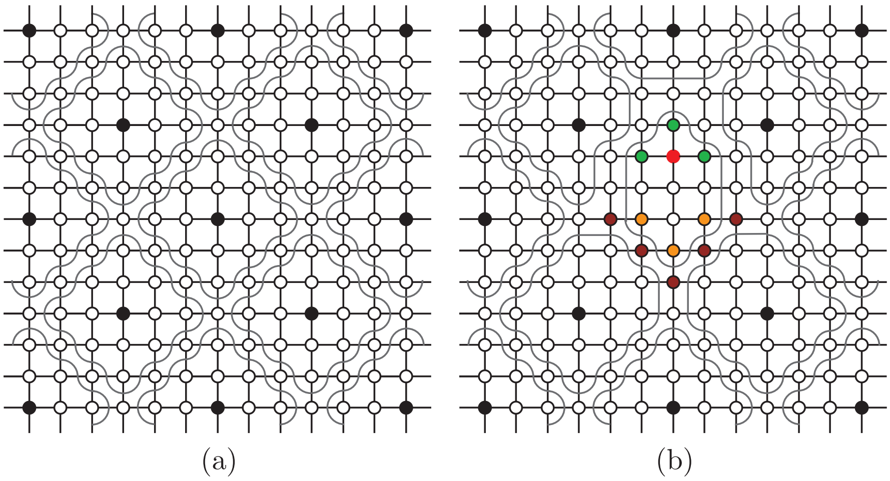


Figure 3: Grid topology. (a) Social Optimum. (b) Non-optimal, irregular arrangement. Green nodes have lower costs, orange nodes higher costs compared to the regular arrangement.

4.3 Grid

4.3.1 Social Optimum

Lemma 4.5 (Optimal Arrangement)**.** *The social cost of an arrangement of caching nodes is minimized if the objects are placed at a regular distance d.*

In the following we outline a proof of Lemma 4.5. In Figure 3 nodes are colored which experience a change in their social cost compared to an regular arrangement. The set N_g contains the green nodes, which have a lower social cost than in a regular distribution, $\sum_{i \in N_g} irregularCost_i < \sum_{i \in N_g} cost_i$. Accordingly, the set of orange and dark-orange nodes N_o experience a higher social cost $\sum_{i \in N_o} irregularCost_i > \sum_{i \in N_o} cost_i$. The five dark-orange nodes account for the increase in the social cost due to the irregularity. It remains to be shown that this local increase can be generalized to the whole grid. In the following we conduct our calculations with a regular distribution of objects.

Lemma 4.6 (Social Optimum)**.** *The social cost of the Social Optimum in the grid for $1 < \alpha < n$ is*

$$Cost_{OPT} \in \Theta(\alpha^{1/3} \cdot n).$$

PROOF. We prove the upper and lower bound in turn.

Lower Bound. Consider an arbitrary placement of objects at equal distances d in both dimensions (see Figure 3(a)). We assume $d \gg d_{opt}$ at the beginning

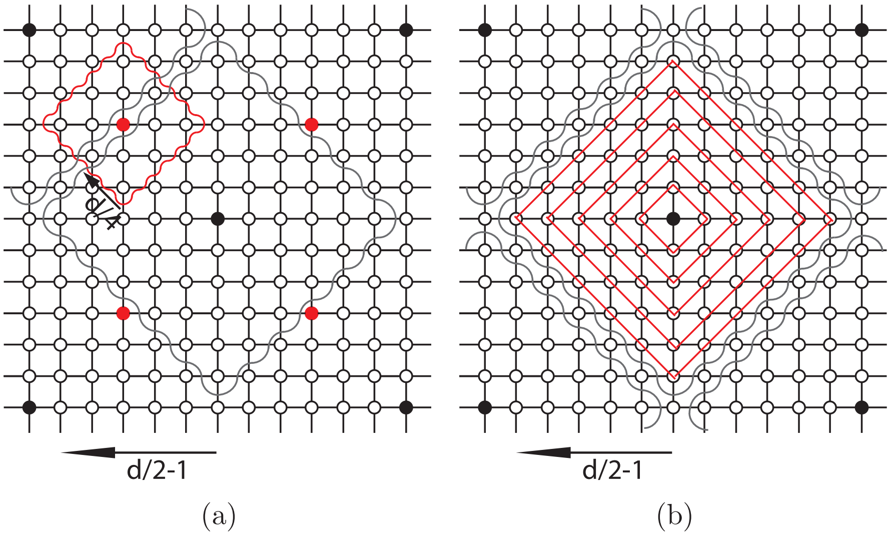


Figure 4: Calculation of social cost. (a) Social Optimum. The red nodes cause $Cost_{NEW}$, the grey ones cause $Cost_{OLD}$. (b) Grey is shown the influence areas of the caching nodes. The red rectangles connect nodes with equal distances to the central caching node.

and try to lower the social cost by placing additional objects in the middle. The cost reduction has to be at least α, otherwise we can not further improve the social cost. Compare Figure 4(a) for the calculation.

$$CostReduction = Cost_{OLD} - Cost_{NEW} = 2\sum_{i=0}^{d/4}((d/4+i)d/2) - 4\sum_{i=0}^{d/4}(i(i+1))$$

If we resolve this equation we obtain: $CostReduction = \frac{7}{96}d^3 - \frac{1}{6}d$. It holds, $distReduction \in \Omega(d^3)$. Thus the distance between cached copies in the *Social Optimum* is $d \in \Omega(\alpha^{1/3})$. To calculate the total social cost of the *Social Optimum* we sum up the costs occurring in all influence areas. The fringe effect of nodes in between influence areas, see Figure 4(b), can be neglected for α not too small, $\alpha > 3$.

$$Cost_{OPT}(d) = \underbrace{\alpha \cdot |I|}_{caching\ cost} + \underbrace{\sum_{I}(\sum_{i \in I_a} d_{i,a})}_{accessing\ cost} \quad (9)$$

In Figure 4(b) nodes which have same distances to the cached copy are connected by rectangles. Therefore we can calculate the social cost

$$
\begin{aligned}
Cost_{OPT}(d) &= \alpha \cdot \frac{n}{d^2/2} + \frac{n}{d^2/2} \cdot 4 \sum_{i=1}^{d/2-1} i^2 && (10)\\
&= \alpha \cdot \frac{n}{d^2/2} + \frac{n}{d^2/2} \cdot \left(\frac{d^3}{6} - \frac{d^2}{2} + \frac{d}{3}\right) && (11)\\
&= \alpha \cdot \frac{n}{d^2/2} + n \cdot \left(\frac{d}{3} - 1 - \frac{2}{3d}\right) &&\\
&= \alpha \cdot \frac{n}{d^2/2} + \frac{1}{3} \cdot n \cdot d && (12)
\end{aligned}
$$

If we insert the lower bound for the distance $d = \alpha^{1/3}$ in (12) we obtain:

$$
Cost_{OPT} = 2\alpha^{1/3} \cdot n + \frac{1}{3}\alpha^{1/3} \cdot n \in \Omega(\alpha^{1/3} \cdot n)
$$

Therefore the social cost of the *Social Optimum* is lower bounded by $Cost_{OPT} \in \Omega(\alpha^{1/3} \cdot n)$. □

Upper Bound. We obtain a example for the social optimum if we oppose the *Caching Cost* to the *Accessing Cost* in (9). From (11) we obtain $\alpha = d^3/6 - d^2/2 + d/3$. An approach to the exact solution obtained with Maple is $d = 2 \cdot \alpha^{1/3}$. Inserting this value in (3) leads us to the social cost and establishes the upper bound of the *Social Optimum.*

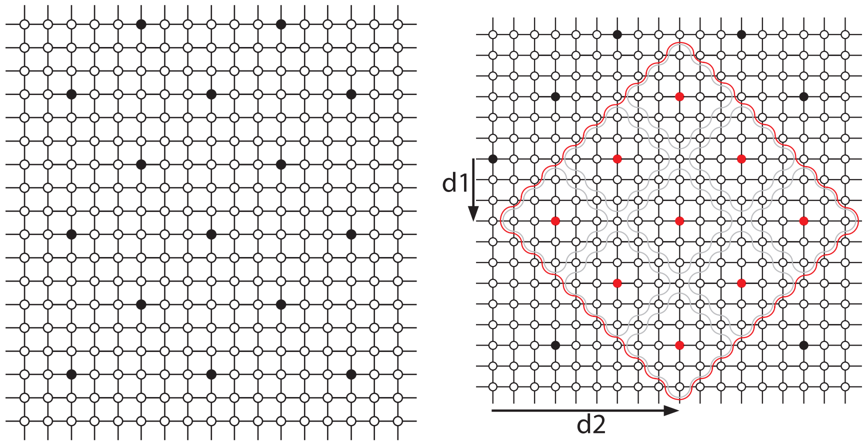
$$
\begin{aligned}
Cost_{OPT}(d) &= \frac{1}{2}\alpha^{1/3}n + \frac{1}{3}\alpha^{1/3}n - 2\alpha^{2/3} + \frac{2}{3}\alpha^{1/3}\\
&\in O(\alpha^{1/3} \cdot n)
\end{aligned}
$$

□

4.3.2 Nash Equilibrium

Lemma 4.7 (Nash Equilibrium)**.** *For the distances between two neighboring caching nodes i, j it holds $\alpha < d_{i,j} \leq 2 \cdot \alpha$ in a Nash equilibrium.*

Proof. If the distance $d_{i,j}$ between the cached copies at nodes i, j is smaller or equal α, then either node i or node j changes its strategy and accesses a remote copy to lower its individual cost. If $d_{i,j}$ is bigger than 2α then a caching node would again change its strategy to lower its individual cost. Therefore a Nash equilibrium only exists, if all distances between two neighboring caching nodes i, j lie in $\alpha < d_{i,j} \leq 2\alpha$. □



(a) (b)

Figure 5: Grid topology for $\alpha = 3$. (a) Nash equilibrium causing high social cost. (b) Worst case arrangement of Byzantine nodes in the oblivious model for $b = 9$. Grey is shown the influence area of one Byzantine node, red is shown the influence area of all 9 Byzantine nodes. $d1 = \alpha, d2 = \sqrt{b}\alpha$.

Lemma 4.8 (Social Cost of the Nash Equilibrium)**.** *The social cost of the worst case Nash equilibrium for* $1 < \alpha < n$ *is*

$$Cost_{NE} \in \Theta(\alpha \cdot n).$$

PROOF. We prove the upper and lower bound in turn.

Upper Bound. The social cost of a Nash equilibrium is $Cost_{NE}$ =*Caching Cost*+*Accessing Cost*. Trivially, it holds that the *Caching Cost* cannot exceed $\alpha \cdot n$. The worst case is reached if all nodes cache a copy. According to Lemma 4.4 the distances in a Nash equilibrium cannot exceed 2α. Therefore the *Accessing Cost* cannot exceed $2\alpha \cdot n$, even if all nodes access a cached copy at maximum distance. Thus the social cost of the worst case Nash equilibrium is upper bounded by $Cost_{NE} \in O(\alpha \cdot n)$.

□

Lower Bound. We give an example for a Nash equilibrium causing high social cost to establish the lower bound. If we assume that the objects are cached at regular distances d then the social cost is

$$
\begin{aligned}
Cost_{NE}(d) &= \alpha \cdot \frac{n}{d^2/2} + \frac{n}{d^2/2} \cdot 4 \sum_{i=1}^{d/2-1} i^2 && (13)\\
&= \alpha \cdot \frac{n}{d^2/2} + \frac{n}{d^2/2} \cdot \left(\frac{d^3}{6} - \frac{d^2}{2} + \frac{d}{3}\right) && (14)\\
&= \alpha \cdot \frac{n}{d^2/2} + n \cdot \left(\frac{d}{3} - 1 - \frac{2}{3d}\right)\\
&= \alpha \cdot \frac{n}{d^2/2} + \frac{1}{3} \cdot n \cdot d && (15)
\end{aligned}
$$

By increasing distance d we increase (15) for $\alpha \geq 3$. According to Lemma 4.7 d is bounded to the interval $[\alpha, 2\alpha]$. Thus we get an cost-intensive example for a Nash equilibrium by setting $d = 2\alpha$. The invoked social cost then is $Cost_{NE} = \frac{n}{2\alpha} + \frac{2}{3} \cdot n \cdot \alpha$ and establishes the lower bound of the worst case Nash equilibrium, $Cost_{NE} \in \Omega(\alpha \cdot n)$.

□

4.3.3 Oblivious Model, Byzantine Nash Equilibrium

Lemma 4.9 (Byzantine Nash Equilibrium)**.** *The social cost of the Byzantine Nash Equilibrium in the grid for* $1 < \alpha < n$, $0 \leq b < \frac{n}{2\alpha^2}$ *is*

$$Cost_{BNE} \in \Theta(\alpha \cdot s + \alpha^3 \cdot b \cdot \sqrt{b}).$$

PROOF. We prove the upper and lower bound in turn.

Lower Bound. As we have already used before the objects are cached regularly every 2α in both dimensions for a Nash equilibrium causing high social cost. In Figure 5(b) you can see a possible worst case arrangement for the Byzantine nodes. The nodes inside this Byzantine area have to access a cached copy outlying, which causes the additional cost

$$AddCost_{I_{b_{max}}} = 4 \cdot \textstyle\sum_{i=1}^{\alpha\sqrt{b}-1} \Big(\underbrace{(\alpha + \alpha\sqrt{b}}_{\substack{\textit{offset to}\\ \textit{midpoint}}} - i) \cdot i \Big)$$

To facilitate calculation we substitute $z = \alpha\sqrt{b}$

$$
\begin{aligned}
AddCost_{I_{b_{max}}} &= 4\Big((\alpha + z)\sum_{1}^{z-1} i - \sum_{1}^{z-1} i^2\Big) \\
&= 4\Big((\alpha + z)\cdot\frac{(z-1)z}{2} - \frac{(z-1)z(2z-1)}{6}\Big) \\
&= 2\big((\alpha + z)(z-1)z\big) - \frac{2}{3}\big((z^2 - z)(2z-1)\big) \\
&= \frac{2}{3}z^3 + 2\alpha z^2 - 2\alpha z - \frac{2}{3}z \qquad (16) \\
&= \frac{2}{3}\alpha^3 b\sqrt{b} + 2\alpha^3 b - 2\alpha^2\sqrt{b} - \frac{2}{3}\alpha\sqrt{b} \\
AddCost_{I_{b_{max}}} &\in \Omega(\alpha^3 b\sqrt{b}) \qquad (17)
\end{aligned}
$$

We obtain the total social cost for the Byzantine Nash equilibrium by building the sum of the caching and accessing cost of the selfish nodes s and the caused cost by the Byzantine nodes, $Cost_{BNE} = Caching\ Cost + Accessing\ Cost + AddCost_{I_{b_{max}}}$. The number of selfish nodes lying in the Byzantine area is $\frac{b\cdot\alpha^2}{4} - b$. To be accurate this nodes have to be subtracted from the selfish nodes s contributing to the caching and accessing cost. We neglect this effect in the following and assume $s > \frac{b\cdot\alpha^2}{4} - b$.

$$
Cost_{BNE}(d) = \alpha\cdot\frac{s}{d^2/2} + \frac{s}{d^2/2}\cdot 4\sum_{i=1}^{d/2-1} i^2 + AddCost_{I_{b_{max}}} \qquad (18)
$$

Inserting the results from (15) and (17) in (18) we get

$$
= \frac{s}{\alpha} + \frac{4}{3}\alpha\cdot s - 2s - \frac{2s}{3\alpha} + \frac{2}{3}\alpha^3 b\sqrt{b} + 2\alpha^3 b - 2\alpha^2\sqrt{b} - \frac{2}{3}\alpha\sqrt{b}
$$

Therefore the Byzantine Nash equilibrium is lower bounded by $Cost_{BNE}(d) \in \Omega(\alpha\cdot s + \alpha^3\cdot b\cdot\sqrt{b})$

□

Upper Bound. First of all we examine two basic arrangements of Byzantine nodes. We designate the combined Byzantine area of b_i Byzantine nodes as I_{b_i}. In the first arrangement a Byzantine node is surrounded by selfish caching nodes. Here, the selfish nodes in the Byzantine influence area I_{b_1} experience the additional cost $AddCost_{I_{b_1}} = \sum_{i\in I_{b_1}} d_{i,a}$, where the selfish caching node a lies outside of I_{b_1}. This results in the total social cost

$$\begin{aligned}
\textstyle\sum_{I_B} AddCost_{I_{b_1}} &= \sum_{I_B} \sum_{i \in I_{b_1}} d_{i,a} \\
&= b \cdot \sum_{i=1}^{d/2-1} \Big((\alpha + \alpha - i) \cdot i \Big) \\
&= b \cdot \Big(\sum 2\alpha i - \sum i^2 \Big) \\
&= b \cdot \left(\frac{(\alpha-1)\alpha}{2} - \frac{(\alpha-1)\alpha(2\alpha-1)}{6} \right) \\
&= b \cdot \Big(\alpha^3 - \alpha^2 - \frac{1}{6}(2\alpha^3 - 3\alpha^2 + \alpha) \Big) \\
&= \frac{2}{3}\alpha^3 b - \frac{3}{2}\alpha^2 b - \frac{1}{6}\alpha b \\
&\in \Omega(\alpha^3 \cdot b)
\end{aligned}$$

In the second arrangement the Byzantine nodes form a biggest possible Byzantine area $I_{b_{max}}$, compare Figure 5(b). We already calculated the additionally caused cost in (17). We obtain higher cost if we arrange the Byzantine players to one maximal interval $I_{b_{max}}$

$$\begin{aligned}
AddCost_{I_{b_{max}}} &> \sum_{I_B} AddCost_{I_{b_1}} \\
\Omega(\alpha^3 b\sqrt{b}) &> \Omega(\alpha^3 b)
\end{aligned}$$

Now it remains to show that the social cost of any other arrangement $\sum AddCost_{I_{b_i}}$ does not cause higher cost than $AddCost_{I_{b_{max}}}$ by considering the constraint $\sum b_i = b_{max}$. However, it is enough to look at the preceding two cases, because any other arrangement of Byzantine nodes can be classified as one of these two cases. E.g. a line arrangement causes the same cost as the sum of single Byzantine influence areas and the sum of several rectangle arrangement can not exceed the cost of $AddCost_{I_{b_{max}}}$ taking into consideration the constraint. Furthermore any divided Byzantine areas affect fewer nodes than $I_{b_{max}}$ due to fringe effects at the borders of neighboring Byzantine influence areas. Therefore we conclude the cost for a Byzantine Nash equilibrium is maximized if we form on big interval. The arrangement visible in Figure 5(b) maximizes the Byzantine area and gives us the upper bound for the social cost. The calculation was conducted already in (18) and therefore the lower bound equals the upper bound, $Cost_{BNE} \in O(\alpha \cdot s + \alpha^3 \cdot b \cdot \sqrt{b})$.

□

4.3.4 Non-Oblivious Model, Byzantine Nash Equilibrium

Lemma 4.10 (Byzantine Nash Equilibrium)**.** *The social cost of the Byzantine Nash Equilibrium in the grid for* $1 < \alpha < n$*,* $0 \le b < \frac{n}{2\alpha^2}$ *is*

$$Cost_{BNE} \in \Omega\left(\frac{b \cdot s}{\alpha} + \frac{s}{2} + \frac{\alpha^3}{b}\right).$$

PROOF. We prove the lower bound in the following.

Lower Bound. We take a similar approach as already for the line topology. Every player is acting risk-averse and assumes its location to be just at the center of the Byzantine interval I_{Bmax}. This results in objects being cached in smaller intervals $d = 4\alpha/\sqrt{b}$, so that the selfish players can be sure to access a valid copy distant at most 2α. We can calculate the additional caused cost by inserting $z = 4\alpha/\sqrt{b}$ in (16)

$$\begin{aligned}
AddCost_{BNE} &= \frac{2}{3}\left(\frac{4\alpha}{\sqrt{b}}\right)^3 + 2\alpha\left(\frac{4\alpha}{\sqrt{b}}\right)^2 - 2\alpha\left(\frac{4\alpha}{\sqrt{b}}\right) - \frac{2}{3}\left(\frac{4\alpha}{\sqrt{b}}\right) \\
&= \frac{128}{3} \cdot \frac{\alpha^3}{b^2\sqrt{b}} + 8\frac{\alpha^3}{b} - 8\frac{\alpha^2}{\sqrt{b}} - \frac{8}{3} \cdot \frac{\alpha}{\sqrt{b}} \\
AddCost_{BNE} &\in \Omega\left(\frac{\alpha^3}{b}\right)
\end{aligned}$$

According to (18) we get for the social cost for this Byzantine Nash equilibrium

$$\begin{aligned}
Cost_{BNE}(d) &= \alpha \cdot \frac{s}{8\alpha^2/b} + \frac{s}{8\alpha^2/b} \cdot 4 \sum_{i=1}^{2\alpha/\sqrt{b}-1} i^2 + AddCost_{BNE} \\
&= \frac{b \cdot s}{8\alpha} + \frac{b \cdot s}{4\alpha^2} \cdot \frac{(\frac{2\alpha}{\sqrt{b}} - 1)\frac{2\alpha}{\sqrt{b}}}{2} + \Omega\left(\frac{\alpha^3}{b}\right) \\
&= \frac{b \cdot s}{8\alpha} + \frac{s}{2} - \frac{\sqrt{b} \cdot s}{4\alpha} + \Omega\left(\frac{\alpha^3}{b}\right) \\
&\in \Omega\left(\frac{b \cdot s}{\alpha} + \frac{s}{2} + \frac{\alpha^3}{b}\right)
\end{aligned}$$

□

We discuss in the following the Price of Anarchy and the Price of Malice for the grid. The Nash equilibrium has not changed asymptotically with the increased dimension, whereas the cost of the Social Optimum has decreased.

That leads to a higher Price of Anarchy, $PoA = \alpha^{2/3}$. Thus in the grid selfish behavior causes higher damage than in the line topology. In the oblivious model the Price of Malice is increased for $\alpha > \sqrt{b}$

$$
\begin{aligned}
PoM_{OBL-LINE} &\in \Theta\left(1 + \frac{\alpha \cdot b^2}{n}\right) \\
PoM_{OBL-GRID} &\in \Theta\left(1 + \frac{\alpha^2 \cdot b\sqrt{b}}{n}\right)
\end{aligned}
$$

We can se the effect of the increased Byzantine influence area I_{Bmax} by the factor α^2. However, we need quadratic Byzantine players to generate I_{Bmax}, that is why the second factor is decreased in the higher dimension.
The Price of Malice in the non-oblivious model is

$$
\begin{aligned}
PoM_{NON-OBL-LINE} &\in \Omega\left(\frac{b}{\alpha} + \frac{1}{b} + \frac{\alpha}{n}\right) \\
PoM_{NON-OBL-GRID} &\in \Omega\left(\frac{b}{\alpha^2} + \frac{1}{\alpha} + \frac{\alpha^2}{b \cdot n}\right)
\end{aligned}
$$

As we have mentioned already in the discussion of the Price of Malice for the line, it is possible to render $PoM < 1$. In the grid we can construct a similar example with $\alpha = 20, b = 9, n > 1000$. We obtain smaller social cost for the Byzantine Nash equilibirum $Cost_{BNE} = 0.5 \cdot n + 890$ than for the Nash equilibrium $Cost_{NE} = 13.5 \cdot n$. Similar to the line topology, the presence of Byzantine players causes a change in the selfish player's strategy, which results in shrunken caching intervals in the worst case and a $PoM < 1$. Thus the social cost can be lowered under certain conditions.

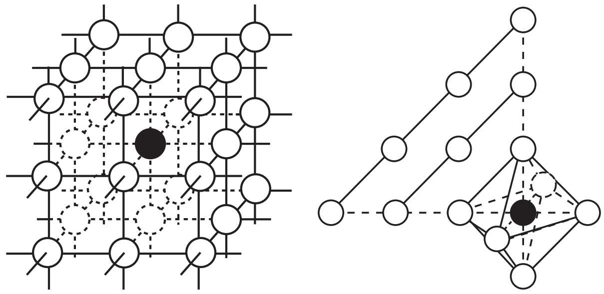


Figure 6: Calculation of Social Cost in 3-Dimensional Grid. (a) 3-dim grid topology. (b) Nodes with equal distances to the center are located on the edges of a octahedron. The edges of the underlying 3-dim grid topology are not shown.

4.4 D-Dimensional Grid

In this chapter we change our notation, here d designates the dimension while we use c for the distance parameter. In Figure 6 a 3-dimensional grid is shown. Nodes with equal distances c to a central node lie on the edges of an octahedron with $radius = c$. The number of vertices $|V_{oct}|$ of such an octahedron is 6 and the number of edges $|E_{oct}|$ is 12. We obtain in a 3-dimensional grid for the number of nodes with equal distances c to a node

$$\begin{aligned} nodes(c) &= |E_{oct}| \cdot (c(c-1)) + |V_{oct}| \cdot c \\ &= 12 \cdot (c(c-1)) + 6 \cdot c \end{aligned}$$

We can easily verify this with Figure 6(b). In the 2-dimensional case we obtain $nodes(c) = 4 \cdot (c(c-1)) + 4 \cdot c$ what corresponds to our findings in the grid. The number of vertices and edges of the polyhedrons connecting equal distant nodes is upper bounded by the values of the underlying grid-topology. In the 3-dimensional grid new edges are introduced for the octahedron, though old ones are skipped and the degree of the nodes does not change. Thus $|E_{oct}|$ and $|V_{oct}|$ is upper bounded to the corresponding numbers of the cube. In higher dimensions we assume that nodes with equal distances again are located on a d-dimensional polyhedron, furthermore its number of edges and vertices is upper bounded the underlying d-dimensional hypercube. These assumptions leads us to the following lemma

Lemma 4.11. *In a d-dimensional grid the number of nodes i with distance $c_{ia} = c$ to node a is*

$$nodes(c) \in O(e \cdot (c(c-1)) + v \cdot c) \tag{19}$$

whereas $e \in O(2^{d-1} \cdot d)$ denotes the number of edges and $v \in O(2^d)$ the number of vertices in the d-dimensional hypercube.

Lemma 4.11 and our results obtained for the line and 2-dimensional grid let us formulate a further Lemma without proving it in this thesis.

Lemma 4.12 (Social Cost)**.** *In a d-dimensional grid where objects are cached at equal distances c the social cost is*

$$\begin{aligned} Cost(d,c) &= \alpha \cdot |I| + |I| \cdot \sum_{i=0}^{c/2-1} nodes(i) \\ &= \alpha \cdot \frac{n}{c^d/d} + \frac{n}{c^d/d} \cdot \sum_{i=0}^{c/2-1} nodes(i) \\ &\in O\left(\alpha \cdot \frac{n}{c^d/d} + \frac{n}{c^d/d} \cdot \sum_{i=0}^{c/2-1} (2^{d-1}d \cdot (i^2 - i) + 2^d \cdot i)\right) \end{aligned} \tag{20}$$

If we compute (20) for the 1-dimensional and 2-dimensional case we obtain

$$Cost(d=1,c) = \alpha \cdot \frac{n}{c/2} + \frac{n}{c/2} \cdot \sum_{i=0}^{c/2-1} 2i \tag{21}$$

$$\begin{aligned} Cost(d=2,c) &= \alpha \cdot \frac{n}{c^2/2} + \frac{n}{c^2/2} \cdot \sum_{i=0}^{c/2-1} (4 \cdot (i^2 - i) + 4i) \\ &= \alpha \cdot \frac{n}{c^2/2} + \frac{n}{c^2/2} \cdot \sum_{i=0}^{c/2-1} 4i^2 \end{aligned} \tag{22}$$

(21) corresponds to the social cost for the line topology which we already have found in (4). Furthermore (22) is equal to the social cost in the grid (10). Thus we have established already the correctness of Lemma 4.12 for $d = 1, d = 2$, however to ensure it holds for higher dimensions further analysis is needed.

5 Conclusion

We have presented the Byzantine Caching Game and analyzed its behavior for different underlying topologies. Therefore we proved tight bounds for the Price of Anarchy and the Price of Malice for the line and the grid topology in the oblivious and non-oblivious model. We have found examples for the non-oblivious model where the presence of Byzantine players improves the worst case social cost.
Furthermore our findings in one and two dimensions have given us indication how they could develop in further dimensions. We formulated two lemmas for the social cost function in d-dimensional grids. A next step would be to prove these and to generalize other findings to d dimensions. The results we obtained in 1-dim and 2-dim for the Price of Malice let us guess that in higher dimensions it develops similarly, $PoM_{OBL} = (1 + \frac{\alpha^d \cdot b^{(d+1)/d}}{n})$. However, the proof of this presumption is beyond the scope of this thesis.

6 Appendix

6.1 Collection of Formulas

$Price\ of\ Anarchy\ PoA = \frac{Cost_{NE}}{Cost_{OPT}}$

$Price\ of\ Byzantine\ Anarchy\ PoBA(b) = \frac{Cost_{BNE}(b)}{Cost_{OPT}}$

$Price\ of\ Malice\ PoM(b) = \frac{PoB(b)}{PoB(0)} = \frac{PoB(b)}{PoA} = \frac{Cost_{BNE}}{Cost_{NE}}$

6.2 Star

SOCIAL OPTIMUM

$\alpha < 1$	$Cost_{OPT} = \alpha \cdot n$	
$1 < \alpha < 2$	$Cost_{OPT} = \alpha + (n-1)$	See figure 1(a).
$\alpha > 2$	$Cost_{OPT} = \alpha + (n-1)$	

WORST NASH EQUILIBRIUM

$\alpha < 1$	$Cost_{NE} = n \cdot \alpha$	Object is cached at every node.
$1 < \alpha < 2$	$Cost_{NE} = (n-1)\alpha + 1$	Object is cached at every leaf node. See figure 1(b).
$\alpha > 2$	$Cost_{NE} = \alpha + 1 + 2(n-2)$	One replica is placed at a leaf node.

PRICE OF ANARCHY

$\alpha < 1$	$PoA = 1$
$1 < \alpha < 2$	$PoA = \frac{(n-1)\alpha+1}{\alpha+(n-1)} \leq \frac{(n-1)2+1}{2+(n-1)} \leq 2$
$\alpha > 2$	$PoA = \frac{\alpha+1+2(n-2)}{\alpha+(n-1)} \leq 2$

BYZANTINE NASH EQUILIBRIUM, OBLIVIOUS MODEL

$\alpha < 1$	$1 \leq b < n$	$Cost_{BNE} = (n - b) \cdot \alpha$ Worst BNE caches object at every server.
	$b = n$	$Cost_{BNE} = 0$ Every Player is byzantine. Nobody has object cached.
$1 < \alpha < 2$	$1 \leq b < n$	$Cost_{BNE} = \infty$ Worst BNE caches just at byzantine center node. See figure 1(c).
	$b = n$	$Cost_{BNE} = 0$
$\alpha > 2$	$1 \leq b \leq n$	$Cost_{BNE} = \infty$ Worst BNE places just one replica at byzantine leaf node.
	$b = n$	$Cost_{BNE} = 0$

PRICE OF BYZANTINE ANARCHY, OBLIVIOUS MODEL

$\alpha < 1$	$1 \leq b < n$	$PoBA = \frac{(n-b)\cdot\alpha}{\alpha+n-1}$
	$b = n$	$PoBA = 0$
$1 < \alpha < 2$	$1 \leq b < n$	$PoBA = \infty$
	$b = n$	$PoBA = 0$
$\alpha > 2$	$1 \leq b \leq n$	$PoBA = \infty$

PRICE OF MALICE, OBLIVIOUS MODEL

$\alpha < 1$	$1 \leq b < n$	$PoM = \frac{n-b}{n}$
	$b = n$	$PoM = 0$
$1 < \alpha < 2$	$1 \leq b \leq n$	$PoM = \infty$
	$b = n$	$PoM = 0$
$\alpha > 2$	$1 \leq b \leq n$	$PoM = \infty$

BYZANTINE NASH EQUILIBRIUM, NON-OBLIVIOUS MODEL

$\alpha < 1$	$1 \leq b < n$	$cost_{BNE} = s \cdot \alpha$ Worst BNE caches object at every server.
	$b = n$	$cost_{BNE} = 0$ Every Player is byzantine.
$1 < \alpha < 2$	$1 \leq b < n-1$	$cost_{BNE} = s \cdot \alpha$ The worst BNE caches replicas at all nodes.
	$b = n$	$cost_{BNE} = 0$
$\alpha > 2$	$1 \leq b < n$	$cost_{BNE} = s \cdot \alpha$
	$b = n$	$cost_{BNE} = 0$

PRICE OF BYZANTINE ANARCHY, NON-OBLIVIOUS MODEL

$\alpha < 1$	$1 \leq b < n$	$PoBA = \frac{s \cdot \alpha}{(n-1)+\alpha}$
	$b = n$	$PoBA = 0$
$1 < \alpha < 2$	$1 \leq b < n-1$	$PoBA = \frac{s \cdot \alpha}{(n-1)+\alpha}$
	$b = n$	$PoBA = 0$
$\alpha > 2$	$1 \leq b < n$	$PoBA = \frac{s \cdot \alpha}{(n-1)+\alpha}$
	$b = n$	$PoBA = 0$

PRICE OF MALICE, NON-OBLIVIOUS MODEL

$\alpha < 1$	$1 \leq b < n$	$PoM = \frac{s \cdot \alpha}{n \cdot \alpha} < 1$
	$b = n$	$PoM = 0$
$1 < \alpha < 2$	$1 \leq b \leq n$	$PoM = \frac{s \cdot \alpha}{(n-1)\alpha+1} < 1$
	$b = n$	$PoM = 0$
$\alpha > 2$	$1 \leq b < n$	$PoM = \frac{s \cdot \alpha}{\alpha+2(n-2)+1}$
	$b = n$	$PoM = 0$

6.3 Line

SOCIAL OPTIMUM

$\alpha < 1$	$Cost_{NE} = \alpha \cdot n$	Social optimum caches object at every server.
$1 < \alpha < n$	$Cost_{OPT} \in \Theta(\sqrt{\alpha} \cdot n)$	Social optimum places objects every $\sqrt{2\alpha}$.
$\alpha > n - 1$	$Cost_{NE} = \alpha + \frac{(n-1)n}{2}$	Social optimum still places replicas every $\sqrt{2\alpha}$.

WORST NASH EQUILIBRIUM

$\alpha < 1$	$Cost_{NE} = n \cdot \alpha$	Worst NE caches object at every server.
$1 < \alpha < n$	$Cost_{NE} \in \Theta(\alpha \cdot n)$	Worst NE places replicas every 2α, see figure 2(a).
$\alpha > n - 1$	$cost_{NE} = \alpha + \frac{(n-1)n}{2}$	Worst NE places one replica at leaf node.

PRICE OF ANARCHY

$\alpha < 1$	$PoA = 1$
$1 < \alpha < n$	$PoA \in \Theta(\sqrt{\alpha})$

BYZANTINE NASH EQUILIBRIUM, OBLIVIOUS MODEL

$\alpha < 1$	$0 \leq b < n$	$cost_{BNE} = s \cdot \alpha$ Worst BNE caches object at every server.
	$b = n$	$cost_{BNE} = 0$ Every Player is byzantine. Nobody has object cached.
$1 < \alpha < n$	$0 \leq b < \frac{n}{2\alpha}$	$cost_{BNE} = \Theta(\alpha \cdot s + \alpha^2 b^2)$ Worst BNE places replicas every 2α, the Byzantine Players are successively located every 2α. See figure 2(c).
	$b = \frac{n}{2\alpha}$	$cost_{BNE} = \infty$ Every player who pretends to cache is byzantine. Nobody has object cached.
$\alpha > n - 1$	$b \geq 1$	$cost_{BNE} = \infty$ Worst BNE places one replica at leaf node. Because $b \geq 1$ nobody caches a copie.
	$b = n$	$cost_{BNE} = \infty$

PRICE OF BYZANTINE ANARCHY, OBLIVIOUS MODEL

$\alpha < 1$	$0 \leq b < n$	$PoBA = \frac{s}{n}$
	$b = n$	$PoBA = 0$
$1 < \alpha < n$	$0 \leq b < \frac{n}{2\alpha}$	$PoBA = \frac{\Theta(\alpha \cdot s + \alpha^2 b^2)}{\Theta(\sqrt{\alpha} \cdot n)} = \Theta(\sqrt{\alpha} + \frac{\alpha^{3/2} b^2}{n})$
	$b = \frac{n}{2\alpha}$	$PoBA = \infty$
$\alpha > n - 1$	$b \geq 1$	$PoBA = \infty$

PRICE OF MALICE, OBLIVIOUS MODEL

$\alpha < 1$	$0 \le b < n$ $b = n$	$PoM = \frac{s}{n}$ $PoM = 0$
$1 < \alpha < n$	$0 \le b < \frac{n}{2\alpha}$ $b = \frac{n}{2\alpha}$	$PoM = \frac{\frac{\Theta(\alpha \cdot s + \alpha^2 b^2)}{\Theta(\sqrt{\alpha} \cdot n)}}{\frac{\Theta(\alpha \cdot n)}{\Theta(\sqrt{\alpha} \cdot n)}} = \Theta(1 + \frac{\alpha \cdot b^2}{n})$ $PoM = \infty$
$\alpha > n - 1$	$b \ge 1$	$PoM = \infty$

BYZANTINE NASH EQUILIBRIUM, NON-OBLIVIOUS MODEL

$\alpha < 1$	$0 \le b < n$ $b = n$	$cost_{BNE} = s \cdot \alpha$ Worst BNE caches object at every server. $cost_{BNE} = 0$ Every Player is byzantine. Nobody has object cached.
$1 < \alpha < n$	$0 \le b < \frac{n}{\frac{4\alpha}{b}}$	$Cost_{BNE} \in \Omega(b \cdot s + \frac{\alpha \cdot s}{b} + \alpha^2)$ The intervals between two cached copies are of size $\frac{4\alpha}{b}$. If a player supposes b players are byzantine around him, he experiences again a size of 2α.

PRICE OF BYZANTINE ANARCHY, NON-OBLIVIOUS MODEL

$\alpha < 1$	$0 \le b < n$ $b = n$	$PoBA = \frac{s}{n}$ $PoBA = 0$
$1 < \alpha < n$	$0 \le b < \frac{n}{\frac{4\alpha}{b}}$	$PoBA = \frac{\Omega(b \cdot s + \frac{\alpha \cdot s}{b} + \alpha^2)}{\Theta(\sqrt{\alpha} \cdot n)}$

PRICE OF MALICE, NON-OBLIVIOUS MODEL

$\alpha < 1$	$0 \le b < n$ $b = n$	$PoM = \frac{s}{n}$ $PoM = 0$
$1 < \alpha < n$	$0 \le b < \frac{n}{\frac{4\alpha}{b}}$	$PoM = \frac{\Omega(b \cdot s + \frac{\alpha \cdot s}{b} + \alpha^2)}{\Theta(\alpha \cdot n)} = \Omega(\frac{b}{\alpha} + \frac{1}{b} + \frac{\alpha}{n})$

6.4 Grid

SOCIAL OPTIMUM

$\alpha < 1$	$Cost_{OPT} = \alpha \cdot n$
$1 < \alpha < n$	$Cost_{OPT} \in \Theta(\alpha^{1/3} \cdot n)$

WORST NASH EQUILIBRIUM

$\alpha < 1$	$Cost_{NE} = \alpha \cdot n$	Worst NE caches object at every server. Social optimum as well.
$1 < \alpha < n$	$Cost_{NE} = \Theta(\alpha \cdot n)$	Worst NE places objects in both dimensions every 2α .

PRICE OF ANARCHY

$\alpha < 1$	$PoA = 1$
$1 < \alpha < n$	$PoA = \frac{\Theta(\alpha \cdot n)}{\Theta(\alpha^{1/3} \cdot n)} = \alpha^{2/3}$

BYZANTINE NASH EQUILIBRIUM, OBLIVIOUS MODEL

$\alpha < 1$	$0 \leq b < n$	$cost_{BNE} = s \cdot \alpha$ Worst BNE caches object at every server. Social optimum as well.
	$b = n$	$cost_{BNE} = 0$ Every Player is byzantine. Nobody has object cached.
$1 < \alpha < n$	$0 \leq b < \frac{n}{2\alpha^2}$	$cost_{BNE} = \Theta(\alpha \cdot s + \alpha^3 \cdot b \cdot \sqrt{b})$ Worst BNE places replicas every 2α in both dimensions. The Byzantine Players form a largest possible "byzantine" area (compare drawings).
	$b \geq \frac{n}{2\alpha^2}$	$cost_{BNE} = \infty$ Every player who pretends to cache is byzantine. Nobody has object cached.
$\alpha > n - 1$	$b \geq 1$	$cost_{BNE} = \infty$ Worst BNE places one replica at leaf node. Because $b \geq 1$ nobody caches a copie.

PRICE OF BYZANTINE ANARCHY, OBLIVIOUS MODEL

$\alpha < 1$	$0 \leq b < n$	$PoBA = \frac{s}{n}$
	$b = n$	$PoBA = 0$
$1 < \alpha < n$	$0 \leq b < \frac{n}{2\alpha^2}$	$PoBA = \frac{\Theta(\alpha \cdot s + \alpha^3 \cdot b \cdot \sqrt{b})}{\Theta(\alpha^{\frac{1}{3}} \cdot n)} \in \Theta(\alpha^{2/3} + \frac{\alpha^{8/3} b \sqrt{b}}{n})$ To compare: $PoBA_{LINE} \in \Theta(\sqrt{\alpha} + \frac{\alpha^{3/2} b^2}{n})$
	$b \geq \frac{n}{2\alpha^2}$	$PoBA = \infty$
$\alpha > n - 1$	$b \geq 1$	$PoBA = \infty$

PRICE OF MALICE, OBLIVIOUS MODEL

$\alpha < 1$	$0 \leq b < n$	$PoM = \frac{s}{n}$
	$b = n$	$PoM = 0$
$1 < \alpha < n$	$0 \leq b < \frac{n}{2\alpha^2}$	$PoM = \frac{\Theta(\alpha \cdot s + \alpha^3 \cdot b \cdot \sqrt{b})}{\Theta(\alpha \cdot n)} \in \Theta(1 + \frac{\alpha^2 \cdot b\sqrt{b}}{n})$ To compare:$PoM_{LINE} \in \Theta(1 + \frac{\alpha \cdot b^2}{n})$
	$b \geq \frac{n}{2\alpha^2}$	$PoM = \infty$
$\alpha > n - 1$	$b \geq 1$	$PoM = \infty$

BYZANTINE NASH EQUILIBRIUM, NON-OBLIVIOUS MODEL

$\alpha < 1$	$0 \leq b < n$	$cost_{BNE} = s \cdot \alpha$ Worst BNE caches object at every server. Social optimum as well.
	$b = n$	$cost_{BNE} = 0$ Every Player is byzantine. Nobody has object cached.
$1 < \alpha < n$	$0 \leq b < \frac{n}{2\alpha^2}$	$Cost_{BNE} \in \Omega\left(\frac{b \cdot s}{\alpha} + \frac{s}{2} + \frac{\alpha^3}{b}\right)$ $d = \frac{4\alpha}{\sqrt{b}}$
	$b \geq \frac{n}{2\alpha^2}$	$cost_{BNE} = \infty$ Every player who pretends to cache is Byzantine. Nobody has object cached.
$\alpha > n - 1$	$b \geq 1$	$cost_{BNE} = \infty$ Worst BNE places one replica at leaf node. Because $b \geq 1$ nobody caches a copie. Social optimum still places replicas every $\sqrt{2\alpha}$.
	$b = n$	$cost_{BNE} = \infty$

PRICE OF BYZANTINE ANARCHY, NON-OBLIVIOUS MODEL

$\alpha < 1$	$0 \leq b < n$ $b = n$	$PoBA = \frac{s}{n}$ $PoBA = 0$
$1 < \alpha < n$	$0 \leq b < \frac{n}{2\alpha^2}$ $b \geq \frac{n}{2\alpha^2}$	$PoBA = \frac{\Omega(\frac{b \cdot s}{\alpha} + \frac{s}{2} + \frac{\alpha^3}{b})}{\Theta(\alpha^{1/3} \cdot n)} \in \Omega(\frac{b}{\alpha^{4/3}} + \frac{\alpha^{8/3}}{b \cdot n})$ To compare: $PoBA_{LINE} \in \Omega(\sqrt{\alpha} + \frac{\alpha^{3/2} b^2}{n})$ $PoBA = \infty$
$\alpha > n - 1$	$b \geq 1$	$PoBA = \infty$

PRICE OF MALICE, NON-OBLIVIOUS MODEL

$\alpha < 1$	$0 \leq b < n$ $b = n$	$PoM = \frac{s}{n}$ $PoM = 0$
$1 < \alpha < n$	$0 \leq b < \frac{n}{2\alpha^2}$ $b \geq \frac{n}{2\alpha^2}$	$PoM = \frac{\Omega(\frac{b \cdot s}{\alpha} + \frac{s}{2} + \frac{\alpha^3}{b})}{\Theta(\alpha \cdot n)} \in \Omega(\frac{b}{\alpha^2} + \frac{1}{\alpha} + \frac{\alpha^2}{b \cdot n})$ $PoM_{LINE-NON-OBL} \in \Omega(\frac{b}{\alpha} + \frac{1}{b} + \frac{\alpha}{n})$ $PoM_{GRID-OBL} \in \Theta(1 + \frac{\alpha^2 \cdot b\sqrt{b}}{n})$ $PoM = \infty$
$\alpha > n - 1$	$b \geq 1$	$PoM = \infty$

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