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Master Thesis

Word of Mouth

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First of all I especially would like to thank my parents for their great support during my studies. Then I have to thank my brothers and friends which supported me greatly the last few months. Furthermore I must thank my advisor and my supervising professor who were responsible for a very beneficial research atmosphere and provided a great support.

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Abstract

The dissemination of rumors in a network is modeled as a game. There, the choice of most suitable starting nodes is one important aspect. Analyzes with different models of the rumor game in various network topologies are performed. For small world networks and random graphs it is simulated using a network algorithm tool. Furthermore concepts of Voting theory are used to gain a broader insight in the analysis of this rumor game.

Contents

1 Introduction

Rumors can spread astoundingly fast through networks of people. Traditionally this happens by word of mouth, but with the emergence of the internet and its new possibilities new ways of rumor propagation are present. People write email, chat with each other or publish their thoughts in a blog. In these networks many factors influence the dissemination of rumors. It is important who initiates at which place some piece of information and how convincing it is. Furthermore the underlying network structure decides how fast the information can propagate and how many people can be reached.

More generally we can speak of diffusion of information in networks. This was the topic of various research during the past years. The analysis of these diffusion processes can be useful for viral marketing, e.g. to target a few influential people for initating marketing campaigns. How the diffusion takes place is tightly influenced by the underlying nework structure.

We aim at modeling the spreading of rumors as a game where a number of players can choose different starting nodes in a graph to spread messages. Additional parameters as the persuasiveness or the aging of a rumor can modulate the propagation of the message through the network. The payoff of each player is the number of nodes that are convinced by the corresponding rumor.

In this thesis we are interested in various aspects of such a rumor game. The choice of nodes that is particularly suitable for initiating the piece of information is one important problem. Of course it is strongly dependent on the underlying network structure. We therefore analyze various network topologies starting with star, line and grid graphs. Furthermore we examine how multiple players influence our rumor game. The players can select from different strategies. In the analysis of the game the existence of Nash equilibria is examined.

Basically we define two different models for the rumor game and perform our examinations in each of them. Moreover we perform simulations in two different small world graph models and in a random graph model. Small world networks have been discovered in various natural phenomena, examples include electric power grids, neural networks, voter networks or social influence networks. Thus our simulations show the behaviour of the rumor game in real world networks and give us further knowledge which rumor placing strategies are beneficial.
The analytical examinations in general graphs are conducted with concepts of facility location theory. After adapting these concepts to our model they give further insights in the problem of choosing the best nodes to initiate the rumor in a general graph.

2
Related Work

In the following we outline three main areas of research related to our rumor game.

2.1 Diffusion of Information

The determination of an initial set of active nodes that starts the diffusion process is an important task for modeling the diffusion of information. Kleinberg et al. [13] study the optimization problem of selecting the most influential nodes in a social network. The optimal solution is NP-hard, but provable approximation guarantees are provided for efficient algorithms. This *Influence Maximization Problem* asks to find the k-node set of maximum influence, whereas the influence of a set of nodes is the expected number of activated nodes at the end of the diffusion process. Kleinberg et al. achieve an approximation of the optimum by a natural greedy hill-climbing strategy. Their method is better than heuristics based on nodes' degrees and centrality within the network, as well as choosing random nodes. The hill-climbing strategy is always within a factor of at least 63% of the optimal solution for this problem.

Blume [1] studies strategic interaction between players. The players are myopic in their decision making and after they made one, they are locked in for some short period of time. He examines two kinds of strategy revision processes: In *Best-Response Dynamics* each player maximizes instantaneous payoff at each revision opportunity, whereas in *Stochastic-Choice Dynamics* the players choose their strategy from some probability distribution. The strategy revision process is a continuous-time Markov process on the space of configurations and describes the evolution of players' choices through time. The player adapts to the environment in which she plays. The environment is in turn determined by the choices

made by the entire collection of players.
Young [17] considers processes in which new technologies and forms of behaviours are transmitted through social or geographic networks. The player's decisions are based on a combination of their inherent payoff and the number of neighbors who have adopted them. The long-run behavior of such systems is analyzed using a potential function. A change in a player's payoff results in a change in the potential. Every pure Nash equilibrium is a local maximum of the potential function. Young introduces a structural criterion, called close-knittedness. A group is close-knit if its members have a relatively large fraction of their interactions with each other as opposed to outsiders. He shows that when agents have a logistic response function to their neighbor's choices, and they interact in small, close-knit groups, the expected waiting time for diffusion to occur is bounded above independently of the number of agents and independently of the initial state.

2.2 Game-Theoretic Approaches

The propagation of information through a social network has been studied from a game theoretic perspective, in which one postulates an increase in utility for players who adopt the new innovation or learn the new information if enough of their friends have also adopted it.
Morris [15] and Young [17] consider a setting like the following coordination game: in every time step, each node in a social network chooses a type $0, 1$. The players of type 1 have adopted the information. Each player i receives a positive payoff for each of its neighbors that has the same type as i, in addition to an intrinsic benefit that i derives from its type. Each player may have a distinct utility for adopting, depending on his inherent interest in the topic. Suppose that all but a small number of players initially have type 0. Morris and Young explore the question of whether type 1's can "take over" the graph if every node chooses to switch to a type with probability increasing as the number of is neighbors that are of the same type increases.
Morris [15] examines when we get contagion under deterministic best response dynamics in local interaction games. Each player's binary choice in each period is best response to the population choices of the previous period. In this setting maximal contagion occurs when local interaction is sufficiently uniform and there is low neighbor growth. The game model consists of a set of players, furthermore it is specified which players interact with which other players. The players interact with a finite subset of the population. Each player at each location has a set of available actions and a payoff function. A player chooses one of two actions to play against all neighbors.
Ellison [3] examines the dynamic implications of learning in a large population coordination game. In each period of a dynamic model the players are randomly matched and each pair plays a 2×2 coordination game. There exist two matching rules, uniform and local. The game is repeated till it converges. Again the players

react myopically to their environment.
In our rumor game we concentrate on the decision making at the beginning, that is when the players are choosing their nodes to initate the rumor in the network.

2.3 Location Problems

In the one-dimensional space Hotelling [11] examines a competitive location problem. He analyzes the establishement of ice-cream shops at a long beach where the customers are distributed uniformly and buy their ice-creams at the nearest shop.
Voronoi Games are used to process the problem in two dimensions. In these games the location set is continuous, and the consumers are assumed to be uniformly distributed. Cheong et al. [2] suppose that the Voronoi Game is played on a square with uniform demand and with a large enough number of moves. The second player locates all her points after observing all of player 1's moves. They show that in this setting player 2 obtains a payoff of at least $1/2+\alpha$ for a fixed constant α. In our model the location set is discrete, thus we cannot just adopt these calculations.
In the competitive location model of Hakimi [9] two competitors alternately choose locations for their facilities in the plane. Here the leader takes into consideration the reaction of the follower by choosing her positions. The follower has full knowledge of the leaders choosen positions and correspondly chooses her positions. Hakimi shows that finding the leader's and the follower's position on general graphs is NP-hard. Our model differs from Hakimi's in various aspects. Hakimi enables the placing of users on edges, and the placing of multiple users at vertices. In our model each vertex can be interpreted as one user.

3
Preliminaries

In this section a model for the propagation of the rumors in the network is described. Furthermore two different models of the rumor game are defined and the topology model is described.

3.1 Propagation

Our *Flooding Model* defines the propagation of the rumors in a graph $G(V, E)$. Two rumors with persuasiveness psv_i are initiated at starting nodes $s_i \in V$. The propagation of the rumors stops if all nodes have received one rumor.
In round k node i believes rumor r_i with probability $P_i = \frac{\sum_{m_i} psv_i(m_i)}{\sum_m psv_i(m)}$, where m_i is a message received from a neighbor containing rumor r_i. Thus P_i depends on the rumors obtained in round $k-1$. Rumor r_i is then propagated in round k to all neighbors which have not yet received one.
Granovetter [8] and Schelling [16] were among the first to define a model that handles the propagation of information in networks by introducing the *Threshold Model*. There, the information is propagated to all neighbors if the summed up persuasiveness of the received messages exceeds a threshold, $\sum_m pers(m) \geq t$.
A further basic diffusion model is the *Independent Cascade Model*, recently investigated in the context of marketing by Goldenberg, Libai and Muller [6] [7]. In this model node i is given the single chance to propagate rumor r_i to neighbor j with probability $p_{i,j}$. There are no further attempts of node i to activate node j. This process runs till no further activations are possible.

3.2 Rumor Game

In this section two models are described that are used for the selection of the starting nodes of the rumor game, the *Basic Model* and the *Bidding Model*.

3.2.1 Basic Model

Consider two players p_1, p_2 and a graph $G(V, E)$. Vertices $v_i \in V$ correspond to strategies, edges $e_i \in E$ are interaction possibilities. In the *Basic Model* of the rumor game the players choose their strategy by selecting their starting node in the graph to place a rumor r_i with persuasiveness psv_i. Then the rumors propagate through the graph as specified by the Flooding Model. The payoff for player p_i is calculated when every node has heard a rumor and equals the number of nodes that believe rumor r_i. This model can be extended to multiple players, where each player chooses one node in the graph to initate her rumor.

3.2.2 Bidding Model

The *Bidding Model* is an extension of the Basic Model. It consists of two players, a purse for each player and a graph G(V,E). There is a auction for each node $v \in V$, where the players bid secretely for the node. The highest bid wins the node, the money of the loosing player is lost. Nobody wins the node if bids at a node are equal. The payoff of each player is *payoff= number of convinced nodes*. We can think of different strategies of bidding. A player can choose high degree nodes, choose central nodes or bid for many nodes and distribute these bids uniformly.

3.3 Topology

We perform our analyzes in different regular topologies each of them having n nodes. The *star* contains one central node and $n - 1$ leaves. The *d-dimensional grid (d,l)* has lenght l in each dimension and consists of nodes of degree $2d$ except the nodes at the borders.
Furthermore we perform examinations in *Kleinberg*, respectively *Eppstein small world graphs* and *Eppstein random graphs*. These models are described in detail in Section 4.1.5.

4 Analysis

4.1 Basic Model

We consider the *Basic Model* of the rumor game where player 1 starts placing a rumor with $psv = 1$ at one node, afterwards player 2 places a rumor with $psv = 2$ at another node. If two rumors arrive at the same time at a node then the higher persuasiveness wins, in this case player 2. In the following we perform our examinations for various topologies.

4.1.1 Star

The central node in the star topology dominates all other nodes. The player which places her rumor there isolates the other and wins all remaining nodes. When one player places the rumor at the central node we obtain a Nash equilibrium.

4.1.2 Line

For the analysis of the game we calculate a $n \times n$ payoff table of all possible combinations of strategies. From Table 4.1 we learn which combination of strategies is the best for each player and whether any Nash equilibria exist.
We do not allow that both players choose the same node, therefore the corresponding table entries remain empty. In Table 4.1 there are only non-dominated rows and columns in the table. We obtain a Nash equilibrium if the players choose the two nodes in the middle of the line. In this case no player has an incentive to change its strategy.

Lemma 4.1.1. *In the line topology the winner obtains payoff* $p = \lceil n/2 \rceil$*, the looser* $p = \lfloor n/2 \rfloor$ *in a Nash equilibrium.*

$\mathbf{P1\backslash P2}$	$\mathbf{1}$	$\mathbf{2}$	..	$\mathbf{\frac{n}{2}-1}$	$\mathbf{\frac{n}{2}}$	$\mathbf{\frac{n}{2}+1}$	..	$\mathbf{n-1}$	$\mathbf{n}$
$\mathbf{1}$	-	$n-1$		$\frac{3n}{4}+1$	$\frac{3n}{4}$	$\frac{3n}{4}-1$		$\frac{n}{2}+1$	$\frac{n}{2}$
$\mathbf{2}$	1	-		$\frac{3n}{4}$	$\frac{3n}{4}-1$	$\frac{3n}{4}-2$		$\frac{n}{2}$	$\frac{n}{2}-1$
$\vdots$									
$\mathbf{\frac{n}{2}-1}$	$\frac{n}{4}-1$	$\frac{n}{4}-2$		-	$\frac{n}{2}+1$	$\frac{n}{2}$		$\frac{n}{4}+2$	$\frac{n}{4}+1$
$\mathbf{\frac{n}{2}}$	$\frac{n}{4}$	$\frac{n}{4}-1$		$\frac{n}{2}-1$	-	$\frac{n}{2}-1$		$\frac{n}{4}+1$	$\frac{n}{4}$
$\mathbf{\frac{n}{2}+1}$	$\frac{n}{4}+1$	$\frac{n}{4}$		$\frac{n}{2}-2$	$\frac{n}{2}-1$	-		$\frac{n}{4}$	$\frac{n}{4}-1$
$\vdots$									
$\mathbf{n-1}$	$\frac{n}{2}-1$	$\frac{n}{2}-2$		$\frac{n}{4}$	$\frac{n}{4}+1$	$\frac{n}{4}+2$		-	1
$\mathbf{n}$	$\frac{n}{2}$	$\frac{n}{2}-1$		$\frac{n}{4}+1$	$\frac{n}{4}$	$\frac{n}{4}+1$		$n-1$	-

Table 4.1: Strategies of player 1 are on the left, on the top nodes where player 2 initates her rumor. For each tuple is given the payoff of player 2.

Proof. The two players can be seen as competitors in the Hotelling problem [11]. There, two ice cream shop owners want to locate their shop at the best place at a beach. The customers visit the shop nearest to them. The owners relocate the shops till they are both situated at the middle where they both reach half of the customers. In this situation they do not have any further incentive to change their strategy. This results in a Nash equilibrium with payoff $p = \lceil n/2 \rceil$ for the winner. □

4.1.3 D-dimensional Grid

The construction of a payoff table soon becomes difficult for increasing dimension. However the regularity of a d-dimensional grid let us presume that the Nash equilibrium is still situated in the middle. The middle is defined as the node with minimal maximal distance to any other node.

Lemma 4.1.2. *In a Nash equilibrium the players are choosing adjacent nodes in the middle of a d-dimensional grid.*

Proof. A player p_i places her rumor at a random node i. The opposite player's best response is to place the rumor besides the placed copy in direction to the middle. Thus she ensures a largest possible part of the network for herself by isolating player 1 from that area. However, the first player's best response is now as well to move his copy in direction to the middle to regain the lost area. This procedure continues till both player place their copies in the middle and cannot increase their payoff anymore. □

Observation 4.1.3. *In a d-dimensional grid a Nash equilibrium is always unique.*

Obviously there exist several symmetrical identical solutions of a Nash equilbrium in a d-dimensional grid. We do not consider these as different solutions. In a

regular grid there is only one node with minimal maximal distance to any other node. Therefore the middle is unique and so is the Nash equilibrium as stated in Observation 4.1.3.
Because Lemma 4.1.2 holds and the length l in one dimension is even both players obtain exactly half of the nodes by placing the rumor in the middle. This lets us state following observation.

Observation 4.1.4. *In the 2-dimensional grid topology with even length l every player obtains payoff $p = n/2$ in a Nash equilibrium.*

If the dimension is odd the first player wins $n^{1/d}$ more by placing her rumor in the middle.

Observation 4.1.5. *In the d-dimensional grid topology with odd dimension l the first player obtains payoff $p = n/2 + n^{1/d}$ in a Nash equilibrium.*

4.1.4 Fairness

The Nash equilbira that we have analyzed so far are very different in terms of their fairness. In the following we examine their fairness coefficient.

Definition 4.1.6. *The fairness coefficient of a Nash equilibria in a two player rumor game is*

$$\textit{fairness} = \frac{\textit{payoff player 1}}{\textit{payoff player 2}}.$$

Obviously the star topology is not very fair. If the number of players is $|P| \ll n$, then the payoff for the player who chooses the central node is $\textit{payoff} = n - p$. The corresponding fairness coefficient is very low, $\textit{fairness} = \frac{1}{n-1}$. However, the line topology is very fair, the fairness coefficient being almost 1, $\textit{fairness} = \frac{n/2-1}{n/2+1}$. In a d-dimensional grid one player gains $n^{(d-1)/d}$ more than the other, leading to $\textit{fairness} = \frac{n/2-n^{(d-1)/d}}{n/2+n^{(d-1)/d}}$. Thus for increasing dimension the fairness coefficient is decreasing. If d and n are small enough then the Nash equilibrium can be quite unfair. Compare a 2-dimensional grid with $n = 100$, there the fairness coefficient is 0.6 and therefore not very high. Correspondingly we obtain in a 3-dimensional grid for the same parameters $\textit{fairness} = 0.4$. If n is high then the Nash equilibria is fair, the fairness coefficient being close to 1.0.

4.1.5 Simulation

To simulate the rumor game on various network topologies we use Sinalgo. This is a simulation framework for testing and validating network algorithms. The Jung framework is a further useful tool. It provides a common and extensible language for the modeling, analysis and visualization of data that can be represented as a graph or network. Various network models are implemented. By including the Jung framework in the Sinalgo framework we can simulate the rumor game also on small world graphs and random graphs.

In a simulation in Sinalgo each node executes an algorithm that specifies what is done with an incoming message. Messages can be sent from node to node or can be broadcasted from a node to all neighbors.

The Jung framework implements various models. Most interesting for us are the small world graph models of Kleinberg and Watts. Furthermore we consider also the random graph model of Eppstein.

For the simulation of our rumor game each node processes following simple algorithm. In each round a node i receives messages $m_1, m_2 \in M$ containing $rumor_1$, respectively $rumor_2$. Message m_i is propagated with probability $P_{m_i} = \frac{\sum_{m_i} psv_i(m_i)}{\sum_m psv_i(m)}$ to the neighbors where no message was received from. Node i ignores further messages.

This allows us to simulate our rumor game where each player chooses some node to initate its rumor. In the next sections we first describe the different graph models and then analyze different rankings of nodes in small world graphs.

Kleinberg Small World Model

The Kleinberg graph generator produces a random graph with small world properties. The underlying model is an $n \times n$ toroidal lattice. Each node u has four local connections, one to each of its neighbors, and in addition long range connections. These nodes are chosen randomly according to a probability proportional to $d^{-\alpha}$, where d is the lattice distance between u and its long-range contact v and α is the clustering expononent. Previous examinations have shown that a clustering exponent of 2.0 matches most accurately real small world graphs.

Watts Small World Model

The Watts model defines a small world network using the beta-model as proposed by Duncan Watts. The basic ideas is to start with a one-dimensional ring lattice in which each vertex has k neighbors. Then the edges are randomly rewired with probability β, in such a way that a small world network can be created. Its properties are low characteristic path lengths and a high clustering coefficient.

Eppstein Power Law Graph

Eppstein et al. [4] propose a graph model with power law distribution, what is an important characteristic of web graphs. Faloutsos et al. [5] observed that the internet topology exhibits power law distribution for example in the degree sequence of web graphs. Furthermore power law distribution occurs in epidemiology, population studies, genome distribution and various social phenomena.

HITS Ranking

The hypertext-induced topic selection (HITS) of Kleinberg offers a concept to the ranking of nodes in the network. Nodes are understood as *Hubs* which are connected to other nodes and as *Authorities* that are linked by other nodes. Based on these two terms Kleinberg [14] introduces a ranking for nodes in a network.
In the following we simulate the Basic Model of our two player rumor game. Player 1 chooses a random node, player 2 chooses the node with highest HITS value. We simulate the rumor game 1000 times in a Kleinberg small world graph. In Table 4.2 the results are listed.

Node	Wins	Avg. # won nodes	Avg. HITS value	Avg. degree
Random	18.1%	41%	15.5	12.0
Max. HITS	81.9%	59%	40.5	18.9

Table 4.2: Results of 1000 simulations in the Kleinberg small world model with 625 nodes and clustering exponent 2.0

The number of won nodes is with 59% on average significantly higher. From these simulations we can conclude that choosing the node with the highest HITS value offers great advantage for a player compared to a random node.

Observation 4.1.7. *In a two player rumor game it is significantly better to choose the node with the highest HITS ranking than a random node for the inital placement.*

Varying of Clustering Exponent

The clustering exponent α characterizes the localized degree of the long-range connections in the network. We have the uniform distribution over long-range connections when $\alpha = 0$. As α increases the long-range connections of a node become more and more clustered. Therefore α serves as a structural parameter measuring how connected the nodes in the graph are. We simulate the rumor game in small world graphs in the table below for $0 \leq \alpha \leq 4$. Bigger values of α are not interesting, the probabilty for long-range connections becoming too low.
When α increases the wins and the average won nodes of player 2 decrease. The average maximum HITS value increases whereas the other values like degree and

HITS node \ α	0	1	2	3	4
Wins	97.6%	94.8%	81.9%	73.3%	71.5%
Avg. won nodes	63.5%	62.1%	58.1%	55.5%	53.7%
Avg. max HITS value	33.1	35.0	40.7	49.0	53.7

Table 4.3: Results for player 2 choosing the node with highest HITS value. 1000 simulations for every value of α in a Kleinberg small world graph with 625 nodes.

HITS value of the random node or the degree of the maximum HITS node stay the same. We can conclude that α influences the rumor game, higher values leading to a more balanced outcome of the game.

Observation 4.1.8. *In a Kleinberg small world graph varying the clustering exponent α influences the outcome of the two player rumor game significantly.*

Degree Ranking

The degree of a node is a further measurement to rank nodes. In Table 4.2 we see that the nodes with highest HITS value do also have a significantly higher degree than random nodes. That is not very astonishing, rather does the degree of a node have strong influence on its HITS value.
Again we simulate the rumor game 1000 times in a Kleinberg small world graph. Player 1 chooses a random node and player 2 the node with the highest degree. Compare the following table for the results.

Node	Wins	Avg. won nodes	Avg. HITS value	Avg. degree
Random	9.8%	38.4%	15.9	11.9
Highest degree	90.2%	61.6%	18.9	23.4

Table 4.4: Results of 1000 simulations in the Kleinberg small world model with 625 nodes and clustering exponent 2.0

As we can see player 2 wins now even more games. Furthermore the wins are slightly clearer with 62% won nodes. This lets us state the following observation.

Observation 4.1.9. *In a two player rumor game it is significantly better to choose the node with the highest degree than a random node for the inital placement.*

Comparison of Degree and HITS Ranking

Now it remains to examine which of the two ranking methods is better. For that purpose compare the table below generated by a simulation where player 1 chooses the highest degree node and player 2 the node with the highest HITS value.

Node	Wins	Avg. won nodes	Avg. HITS value	Avg. degree
Highest degree	63.4%	53.1%	18.9	23.3
Highest HITS	36.6%	46.9%	40.9	19.0

Table 4.5: Results of 1000 simulations in the Kleinberg small world model with 625 nodes and clustering exponent 2.0

The dominance of choosing the highest degree node is a bit surprising. However, the wins are quite narrow with only 53% of the nodes for the winning player. Nevertheless we can state following observations.

Observation 4.1.10. *In a two player rumor game it is significantly better to choose the node with the highest degree than the node with highest HITS ranking for the inital placement.*

Observation 4.1.11. *The HITS ranking is not the best measurement for choosing the node to initiate the rumor in the network.*

Watts Small World Model

We want to ensure that the observations do not depend on the Kleinberg model but rather do hold also in other graph models. Therefore we examine the Watts small world graph model in the following. One disadvantage in this model is that the nodes are homogenous in degree, that is most nodes have about the same degree whereas real networks are inhomogenous.

We start with simulating the rumor game two times, player 1 choosing each time a random node and player 2 once the highest degree and once the node with highest HITS value. We observe that choosing the highest degree node or the highest HITS value node is much better than choosing a random node. In each case the players win about 67% of the nodes and 97% of the games, what confirms Observations 4.2.1 and 4.2.3.

In Table 4.6 we compare the HITS ranking and degree ranking for different values of β. The values are noted for the node with the highest HITS value.

The Watts model interpolates between a regular lattice and a random network as β varies. If $\beta = 0$ then the graph equals the basic lattice structure. As β increases the lattice becomes increasingly disordered unitl at $\beta = 1$ we have a random network.

In Table 4.6 we observe that again the first player's strategy dominates when we for once do not consider the special cases $\beta = 0.0$ and $\beta = 1.0$. The variation of β does not have much influence about the results of the rumor game. We have to mention that the results are not that clear in the Watts model the average won nodes beeing close to 50%, however Observations 4.2.4 and 4.2.5 are confirmed once again.

HITS node \ β	0.0	0.2	0.4	0.8	1.0
Wins	87%	37%	38%	43%	30%
Avg. won nodes	49.8%	46.9%	48.6%	48.9%	48.1%
Avg. max HITS value	16.0	39.6	38.9	40.3	41.1
Avg. degree	11.9	18.1	21.1	23.8	24.2

Table 4.6: Results for player 2 choosing the node with highest HITS value. 100 simulations for values of β in a Watts small world graph with 625 nodes and parameter degree $= 6$.

Eppstein Random Graph Model

In this section we examine the outcome of the rumor game in the Eppstein random graph model. It appears that in the Eppstein model the parameters influence the results strongly.

HITS node \ $\#e$	1000	2000	3000	4000	6000
Wins	75%	50%	42%	17%	8%
Avg. won nodes	316	310	298	275	270
Avg. max HITS value	114	50	38	33	29
Avg. degree	18.8	29.5	39.4	48.9	66.7

Table 4.7: Results for player 2 choosing the node with highest HITS value. 100 simulations for every number of edges in a Eppstein random graph with 625 nodes and parameter = 5.0.

In Table 4.7 we can observe that the values are decreasing linearly with increasing number of edges. Only the average degree of the node with highest HITS value is increasing. This is also a logical consequence of the increasing loss of influence of player 2.
The average degree gives us some evidence what choice of parameters could be most realistic. As we have seen in Table 4.2 in the Kleinberg small world model lies this value between 12 and 19 in a graph with 625 nodes. Thus values that are much higher can be interpreted as leading to a degenerated network with too high clustering of the nodes.
Still it remains quite difficult to define what choice of paramteres could be most realistic. However, we can conclude from Table 4.7 that for small number of edges it is better to choose the node with highest HITS value, whereas for high number of edges the highest degree node is better. For random graphs which are similar to small world graphs in terms of their number of nodes and edges we cannot state clearly which strategy is better.

4.2 Basic Model with Multiple Players

In this section we extend the rumor game to multiple players. We look at different cases on the line and grid topology by using the Basic Model. The players want to maximize their payoff and play with a *risk averse* or a *risky strategy*. If a player chooses a risky strategy, she wants to ensure a high maximal payoff. A risk averse strategy wants to secure a high minimal payoff by maximizing the payoff.

4.2.1 Line

Risk Averse Strategy

In the risk averse case, player 1 places the rumor at position $n/6$, compare Figure 4.1. She avoids the middle so that the other players can not isolate her, but she secures a minimal payoff of $n/6$. Player 2 places the rumor similarly at position $5n/6$. Thus player 3 can place her rumor somewhere between the already placed rumors. We assume player 3 to play fair, what lets her choose the middle. This results in a payoff of $n/3$ for every player.
Generalizing the game to k risk averse players, the locating principle stays the same. The rumors are initated at distances $1/k$ starting at $1/(2k)$. Therefore every player obtains *payoff* $= n/k$.

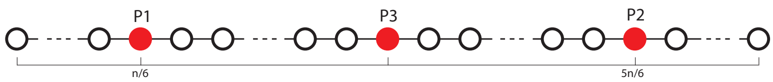

Figure 4.1: Line risk averse. Placing strategies of three players.

Risky Strategy

If the players decide to play with more risk the analysis looks differently. Player 1 and 2 do position their rumors at positions $n/4$ and $3n/4$. Thus they ensure a payoff of $n/4$ and they still have the opportunity to win even $n/2$. For player 3 there remain following possibilities, where she obtains always *payoff* $= n/4$. Compare Figure 4.2 for the three different cases.

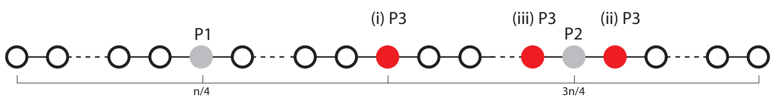

Figure 4.2: Line risky strategy. (i) In this fair case player 3 places the rumor in the middle. (ii) The rumor is placed at $3/4 + 1$. (iii) In the third case player 3 places the rumor at $3/4 - 1$.

Case	Player 1	Player 2	Player 3
(i)	$\frac{3n}{8}$	$\frac{3n}{8}$	$\frac{n}{4}$
(ii)	$\frac{n}{2}$	$\frac{n}{4}$	$\frac{n}{4}$
(iii)	$\frac{n}{4}$	$\frac{n}{2}$	$\frac{n}{4}$

Table 4.8: Lists the payoff of each player in the corresponding case.

From Table 4.10 we learn that player 3 cannot influence his own payoff but decides how much payoff the others obtain. In this case we assume that she chooses the fair strategy, resulting in a *payoff*$=\frac{3n}{8}$ for player 1 and 2.
If we consider k risky and fair players, we observe that player 1 and 2 win slightly more that the others, their payoff being $\frac{1}{k+1}+\frac{1}{2(k+1)}$. They benefit from the fairness of the other players. Each of the remaining players obtains *payoff*$=\frac{1}{k+1}$.

4.2.2 2-dimensional Grid

Risk Averse Strategy

Playing with a risk averse strategy the k players aim at maximizing their payoff and at the same time at ensuring a high minimal payoff. One possible outcome is that the players initate their rumors at nodes located on a circle, see Figure 4.3(a). This results in an equal distribution of the nodes for all players, giving each of them a payoff of n/k.

Risky Strategy

If the players choose more risky strategies, they want to ensure themselves a high maximal playoff. One possible outcome is that they place the rumors on a circle at bigger distances. Thus locating on the circle is not the best place any more for everybody and some players locate the rumor inside the circle. See Figure 4.3 for a possible arrangement.

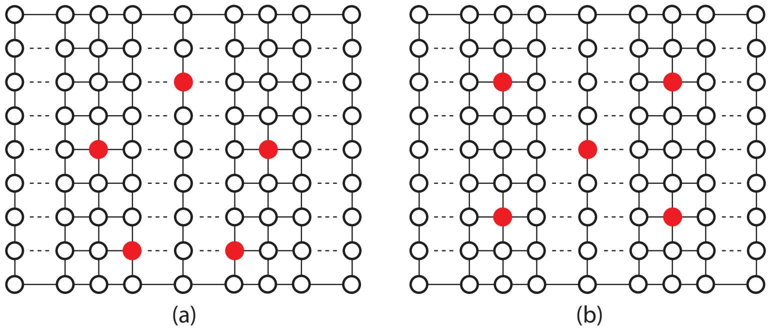

Figure 4.3: 2d grid. (a) Risk averse placing strategies of five players. (b) Risky placing strategies of five players.

4.3 Bidding Model

In the following we analyze the Bidding Model for different topologies. We assume that the players place the rumor at once at the same time. Furthermore it holds for the purse $p \ll n$. It is the player's objective to reach as many nodes as possible. In the following the most central node in network $G = (V, E)$ is the node v_i with minimal maximal distance to any node $v \in V$. The two players can choose between following strategies.

Strategy 1: Bid p for the most central node.
Strategy 2: Choose randomly p nodes and bid 1 for each of them.

We analyze the player's expected payoff for different topologies. Player 1's expected payoff is $E_{P1}[i, j]$ when she is playing strategy i and player 2 is playing strategy j. Obviously the expected payoff of player 2 then is $E_{P2}[i, j] = n - E_{P1}[i, j]$.

4.3.1 Star

As we have seen in Section 4.1.4 the star topology is very unfair in the Basic Model. In the Bidding Model there is again a much bigger payoff for choosing strategy 1 which bids for the central node if $p \ll n$ and both players have the same purse. For the expected payoff of player 1 when both player choose strategy 2 we trivially obtain $E_{P1}[2, 2] = \frac{n}{2}$. If player 1 bids p for the central node, then she wins all nodes except the ones which player 2 has bid for, $E_{P1}[1, 2] = n - p$. Table 4.10 shows the resulting payoffs.

Theorem 4.3.1. *In the star topology the Nash equilibrium is dependent on the value of the purse* p*. For* $p \leq \frac{n}{2}$ *it occurs if both player play strategy* 2*, for* $p > \frac{n}{2}$ *if the two players play different strategies* 1 *and* 2*.*

	P2 Strategy 1	P2 Strategy 2
P1 Strategy 1	$0, 0$	$n - p, p$
P1 Strategy 2	$p, n - p$	$\frac{n}{2}, \frac{n}{2}$

Table 4.9: Star topology. The expected payoffs of both players for the corresponding strategies.

Proof. For $p < \frac{n}{2}$ it holds $n - p > \frac{n}{2}$. In this case the players choose different strategies 1 and 2 by maximizing their payoff. If it holds for their purse $p \geq \frac{n}{2}$ then both players choose strategy 2. □

4.3.2 Line

When both players choose strategy 2 trivially each obtains an expected payoff of $E[2, 2] = n/2$. When the players choose different strategies 1 and 2 the p nodes of player 2's strategy are placed randomly on the line. Therefore $p + 1$ intervals of expected size $\frac{n}{p+1}$ are generated. Player 1's central node is positioned in one of these intervals, thus her expected payoff becomes $E_{P1}[1, 2] = \frac{n}{2(p+1)}$. The following table lists the expected payoffs for both players.

	P2 Strategy 1	P2 Strategy 2
P1 Strategy 1	$0, 0$	$\frac{n}{2p+2}, n - \frac{n}{2p+2}$
P1 Strategy 2	$n - \frac{n}{2p+2}, \frac{n}{2p+2}$	$\frac{n}{2}, \frac{n}{2}$

Table 4.10: Line topology. The expected payoffs of both players for the corresponding strategies.

Theorem 4.3.2. *In the line topology the Nash equilibrium occurs if both players choose strategy* 2.

Proof. We compare the the two expected payoffs of the Nash equilibra. It holds $\frac{n}{2} \geq \frac{n}{2p+2}$ if $p \geq -1$. Therefore if both players choose strategy 2 they do not have any incentive to change their strategy and a Nash equilibirum occurs. □

In Figure 4.4 the expected payoff for changing values of the purse p is showed. For small values of p the expected payoff is high for player 1 who bids for the central node. In this case her opponent's purse allows to bid for few nodes only. For increasing p player 1's payoff is decreasing exponentially. Player 2's purse now enables to spread the rumors to many nodes at the beginning, limiting the influence of the central node strongly.

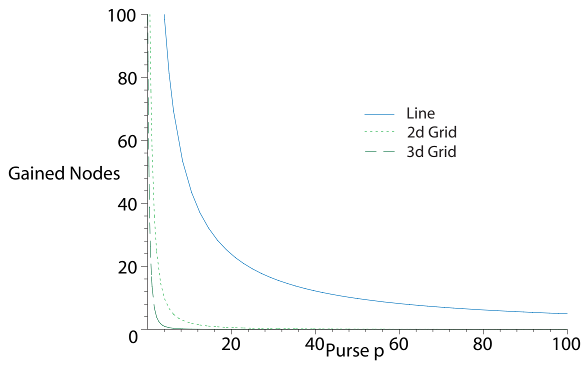

Figure 4.4: Expected payoff of player 1 when she is playing strategy 1 and player 2 is playing strategy 2 for $n = 1000$.

4.3.3 D-dimensional Grid

In the following we generalize the calculation of the expected payoff for d-dimensional grids. For $E_{P1}[2, 2]$ we obtain again an expected payoff of $n/2$.
For the calculation of $E_{P1}[1, 2]$ we project the nodes set by strategy 2 on the d axes. The interval on one axis containing the central node has expected size $\frac{n^{1/d}}{p+1}$. Therefore we obtain for the expected payoff $E_{P1}[1, 2] = \prod_{i=1}^{d} \frac{n^{1/d}}{2(p+1)} = \frac{n}{2^d(p+1)^d}$. Figure 4.4 shows the expected payoffs for different topologies. We observe that the results are similar as already obtained for the line topology. Now, the expected payoff for player 1 is decreasing earlier, because for increasing dimension it needs more nodes to isolate player 1's central node.

	P2 Strategy 1	P2 Strategy 2
P1 Strategy 1	$0, 0$	$\frac{n}{2^d(p+1)^d}, n - \frac{n}{2^d(p+1)^d}$
P1 Strategy 2	$n - \frac{n}{2^d(p+1)^d}, \frac{n}{2^d(p+1)^d}$	$\frac{n}{2}, \frac{n}{2}$

Table 4.11: D-dimensional grid topology. The expected payoffs of both players for the corresponding strategies.

Theorem 4.3.3. *In the d-dimensional grid topology the Nash equilibrium occurs if both player choose strategy* 2.

Proof. The expected payoff $E_{P1}[1, 2] = \frac{n}{2^d(p+1)^d}$ is smaller than $\frac{n}{2}$ for every value of p. Therfore a player has higher payoff when she is playing strategy 2. This results in the Nash equilibrium when both player choose strategy 2. □

5 Location Theory

In location theory it is examined where to locate facilities so that the distance to the users is optimized. We are analyzing the same problem by looking for optimal positions to initate rumors in a network. In our rumor game two players compete for nodes in a graph $G(V, E)$. Each can choose alternately one starting node to place her rumor. Then it is examined how many nodes are reached when the flooding algorithm is started.
Hakimi et al. [9] examine the facility location problem in a weighted graph. The facilities are located at vertices or edges whereas the users are located only at vertices. In their model it is possible to locate multiple users at a vertex. Hakimi et al. develop concepts to determine the best vertices to place the facilities.
We adapt these concepts to our model where only one user is located at each vertex, the set of users U being V. Furthermore the edge weights are restricted to 1 in our graph. In our model we use these concepts to determine the best vertices to start the rumor game. In the following a few terms are introduced and adapted to our model.
The $(r|p)$-medianoid problem in location theory asks to locate r new facilities in the graph which compete with p existing facilities for reaching the most users. Whereas the $(r|p)$-centroid problem examines how to locate first the p facilities when it is known that r facilities are located afterwards by a second player.

Definition 5.0.4. *Finding the* $(r|p)$-medianoid *of network is the problem of player 2 who wants to locate* r *new vertices to initiate the rumor when already* p *vertices are choosen by player 1.*

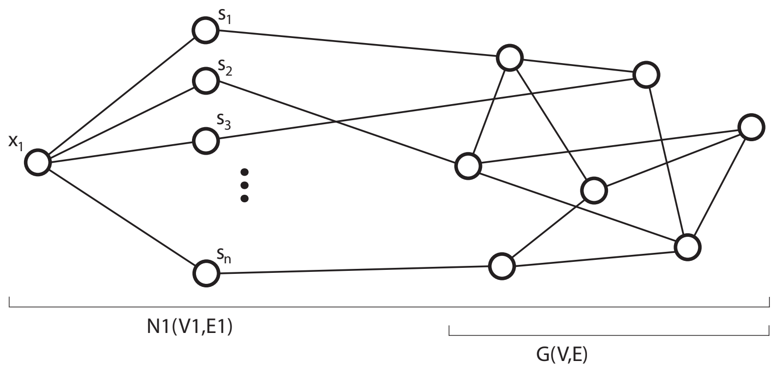

Figure 5.1: Finding the medianoid of a graph.

Definition 5.0.5. *Finding the* $(r|p)$-centroid *of a network is the problem of player 1 who wants to locate optimally p vertices to initiate the rumor when afterwards r rumors are placed by a second player.*

5.1 Finding the Medianoid and Centroid of a Network

In the following we introduce some notations. In graph $G(V, A)$ it holds $D_G(v, Z) = \min\{d(v, z)|z \in Z\}$, where $d(v, z)$ is the length of a shortest path in G from v to z. Thus $D_G(v, Z)$ designates the minimal shortest path from node v to a node $z \in Z$. Let X_p be the set of the p nodes choosen by player 1 and Y_r the set of the r nodes choosen by player 2. The set of vertices that are closer to a rumor published by Y_r than to the ones published by X_p is $V(Y_r|X_p) = \{v \in V|D(v, Y_r) < D_G(v, X_p)\}$. This lets us define the part of the graph controlled by rumors placed at Y_r as $W(Y_r|X_p) = \sum\{w(v)|v \in V(Y_r|X_p)\}$.

Lemma 5.1.1. *The problem of finding an* $(r|X_1)$*-medianoid of a network is NP-hard.*

Proof. We prove this theorem by reducing the dominating set (DS) problem to the $(r|X_1)$-medianoid problem.

For the DS problem there is given a graph $G(V, E)$ with node set V, edges set E and an integer $r < |V|$. It is asked whether there is a subset $V' \in V$ such that $|V'| \leq r$ and $D_G(v, V') \leq 1$ for all $v \in V$.
Given an instance of the DS problem, we construct a network $N_1(V_1, E_1)$ with node set $V_1 = V \cup S \cup x_1$. $|S|$ equals n, a new vertex is introduced for every existing vertex $v \in V$ and is connected to it, compare Figure 5.1. All nodes in

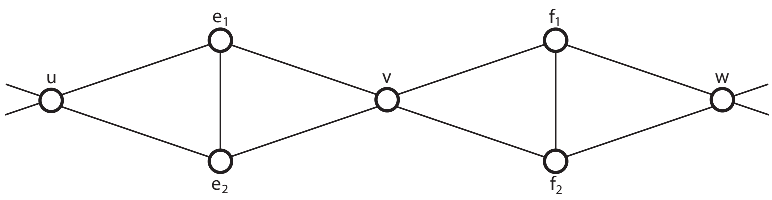

Figure 5.2: Finding the centroid of a graph. Diamond structure in graph N_1.

S are connected to x_1. Thus the edge set is $E_1 = E \cup B \cup T$, whereas $B = \{(s_i, v_i)|s \in S, v \in V\}$ and $T = \{(x_1, s)|s \in S\}$.
We show that there exist r vertices on N_1 composing Y_r such that $W(Y_r|x_1) \geq |V|$, iff the DS problem has a solution. If DS has a solution in G, then there exists $V' \subset V$ with $|V'| = r$ such that $D_G(v, V') \leq 1$ for all $v \in V$. Let $Y_r = V'$. Then it follows $W(Y_r|x_1) = |V| + c$, because $D(v, Y_r) = 1 < d(v, x_1) = 2$. The constant $c = \frac{1}{2}r$ follows from the fact that r nodes lie in the middle of x_1 and some $v_i \in V'$.
Now suppose Y_r is such that $W(Y_r|x_1) \geq |V|$. If $w_i \in Y_r$ it holds $W(w_i|x_1) \leq W(v_j|x_1)$, w_i and v_j being neighbors. This follows from $degree(w_i) = 2$ and the fact that one side is blocked by x_1. Therefore we can move each $w_i \in Y_r$ to its neighbor $v_j \in V$ without loosing value of W. This leads to $Y_r \subset V$. Then it is easy to see that $W(Y_r|x_1) > |V|$, what lets us state for all $v \in V, D(v, Y_r) < d(v, x_1) = 2$. Thus Y_r is a solution to DS. □

Lemma 5.1.2. *The problem of finding an* $(1|p)$*-centroid of a network is NP-hard.*

Proof. We prove this theorem by reducing the vertex cover (VC) problem to the $(1|p)$-centroid problem.

In the VC problem there is given a graph $G(V, E)$ and an integer $p < |V|$. It is asked whether there is a subset $V' \subset V$ with $|V'| \leq p$ such that each edge $e \in E$ has at least one end node in V'.
Given an instance of the VC problem, we construct a network $N_1(V_1, E_1)$ from G by replacing each edge $e_i = (u, v)$ in G by the diamond structure shown in Figure 2.
Let $Y_r(X_p)$ be the set of nodes choosen by player 2 when player 1 has choosen the nodes X_p. We prove the theorem by showing that there exists a set X_p of p vertices on N_1 such that $W(Y_1(X_p)|X_p) \leq 3$ for every vertex $Y_r(X_p)$ on N_1, *iff* VC has a solution.
Suppose V' is a solution to the VC problem in G and $|V'| = p$. Let $X_p = V'$ on N_1. Then for any diamond joining u and v in N_1, either u or v belong to $V' = X_p$. Then it is easy to see that $W(Y_1(X_p)|X_p) \leq 3$ for every point $Y_r(X_p)$ on G.

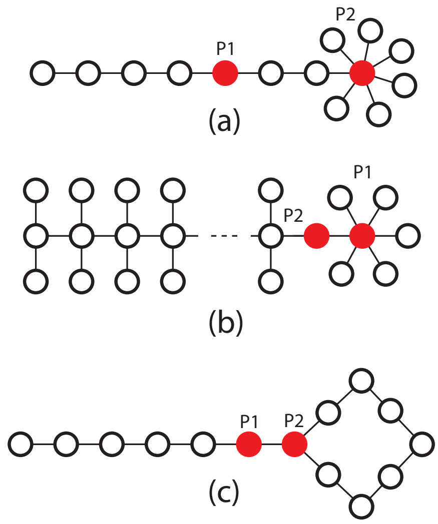

Figure 5.3: Counterexamples for strategies of player 1 where she does not win at least half of the nodes. (a) Player 1 selects the node with smallest radius. (b) She selects the node with highest degree. (c) Player 1 selects the midpoint of the minimum spanning tree. In these examples player 2 always wins at least $n/2$.

On the other hand suppose the set of p points X_p on N_1 is such that $W(Y_1(X_p)|X_p) \leq 3$ for every choice of point $Y_1(X_p)$ on N_1. If on each diamond of N_1 there exists at least one point of X_p, then we can move this point to u or $v \in V' \subset V$. It follows that each diamond has either u or v in V' and therefore V' would provides a solution to the VC problem in G.
Let us assume there is a diamond in N_1 joining u and v on which no point of X_p lies. Suppose $\min\{D(u, X_p), D(v, X_p)\} > 1$ then it is easy to see that $W(e_1|X_p) \geq 4$ what contradicts our presumption. Thus, we may assume that $\min\{D(u, X_p), D(v, X_p)\} = 1$. Now consider another adjacent diamond with exactly one point of X_p at w. Then it holds $0 < D(u, Y_1(X_p)) < D(u, X_p)$, that means $Y_1(X_p)$ lies at f_2, see Figure 2. However, then it follows $W(Y_1(X_p)|X_p) \geq 4$. Thus there have to be at least two points of X_p on the diamond (v, w). In general, if there are no points of X_p on some diamond in N_1, then there are at least two points of X_p on a adjacent diamond of N_1. Therefore, there are enough diamonds to cover all diamonds in N_1, what gives us a solution to VC in G. □

5.1.1 Heuristics for Centroid

The first player can use various strategies to find the optimal node to place the rumor in our two player rumor game. We looked at following strategies for choosing the centroid of a network: choose the node with smallest radius, with largest degree or the midpoint of the minimal spanning tree. However, for these strategies it is easy to find graphs where they do not win, compare Figure 5.3.

In the example shown in Figure 5.3(a) player 1 selects the node v_j with the smallest radius rad_{min}. In this simple case the second player wins more than player 1 by choosing the highest degree node v_i if it holds $degree(v_i) > 3 \cdot rad_{min}/2$. In Figure 5.3(b) player 1 selects the node v_i with highest degree. If it holds $n > 2 \cdot degree(v_i)$ then player 2 wins more than half of the nodes by selecting the neighbor of v_i. When the midpoint of the minimum spanning tree is choosen by player 1 then it is easy to see that player 2 can choose a neighbor and win more than half of the nodes, compare Figure 5.3(c).

5.1.2 Heuristics for Medianoid

In this section we are interested in finding the $(r|r)$-medianoid of a graph. Our approach is to subdivide the graph and to determine in each area the local $(1|1)$-medianoid. By distributing the calculation of the medianoid we obtain an approximation to the optimal solution. Obviously it strongly influences the results how the graph is subdivided. If both players can set about the same number of rumors, $r \approx p$, then we can subdivide the graph by following simple algorithm.

Algorithm BASIC. The clusters S_i of the partition $\mathcal{S}$ are simultaneosly generated. For each of the p nodes choosen by player 1 a cluster S_i is created. Then in each step the cluster is growed by adding layers around it. Each layer contains vertices v_i of constant distance, $d(v_i, p_i) = c$. If two layers meet at the same time at a node then it is assigned with probability $1/2$ to a cluster S_i. The algorithm stops when no further nodes can be added to any cluster S_i.

Each node is reached by this algorithm, therefore the graph is subdivided completely. We obtain a subdivision $\mathcal{S}$ of $G(V, E)$ with following properties.

(1) All $S_i \in \mathcal{S}$ are disjoint.
(2) $\forall v \in V$ it holds $\exists S_i \in \mathcal{S}, v \in S_i$.

The subdivision generated by Algorithm BASIC does not restrict the size of the partitions. One could think that the centrum of the partitions is also the best vertex to choose for a player. Compare Figure 5.3(c) for an example where this vertex is beaten.

In the following player 1 is assumed to place her rumors at the r best positions knowing that a second player is to place another r rumors. In other words we assume that the $(r|r)$-centroid is known to player 1. Consider the partition $\mathcal{S}$ of Algorithm BASIC. In the worst case each S_i corresponds to a star topology where one leaf is shared with each neighbor, resulting in a very unfair outcome of the rumor game for the second player.

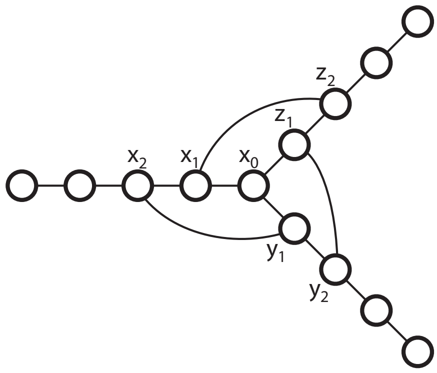

Figure 5.4: Example of a graph where no CondorcetVertex exists.

5.2 CondorcetVertices

For our analyzes of the Rumor Game in different topologies we introduce the *Distance Score* and the *CondorcetVertex*.

Definition 5.2.1. *For any two vertices $v_i, v_j \in V$ the number of vertices that are closer to v_i than to v_j is designated as the* Distance Score*, $DS_i(j) = |\{v \in V : d(v, v_i) < d(v, v_j)\}|$.*

Definition 5.2.2. *A vertex $v_j \in V$ is called a* CondorcetVertex *if $DS_i(j) \leq |V|/2$ for every $v_i \in V$.*

Thus a vertex $v_j \in V$ is called a CondorcetVertex if no more than one half of the vertices gets the rumor from any other vertex in the graph. In a general graph CondorcetVertices must not exist, compare Figure 5.4. In this example for every choosen node of the first player, the second player wins more than $n/2$.
Let $F(x)$ be the sum of shortest paths from a vertex x to all other vertices, $F(x) = \sum_{v \in V} d(v, x)$.

Definition 5.2.3. *A vertex $v_j \in V$ is called* MinSPVertex *if $F(v_j) \leq F(v_i)$ for every $v_i \in V$.*

The MinSPVertex v_j is the vertex which has the minimal sum of shortest paths to all other vertices. In a graph the MinSPVertex does always exist. A CondorcetVertex does not always equal the MinSPVertex, compare Figure 5.5(b) where vertex v_i is a CondorcetVertex but not a MinSPVertex.
CondorcetVertices do not have to be adjacent, compare Figure 5.5(b).
In a complete graph $G_c(V, E)$ every vertex is a CondorcetVertex. It holds for all $v_j \in V$ that $DS_i(j) \leq |V|/2$ for every $v_i \in V$. From G_c we construct a worst case example for the expected payoff of a CondorcetVertex by deleting half of the outgoing edges from one node v_j. The corresponding graph $G'_c V, E$ still consists only of CondorcetVertices. However, vertex v_j with degree $n/2$ has an expected payoff of only $n/4$.

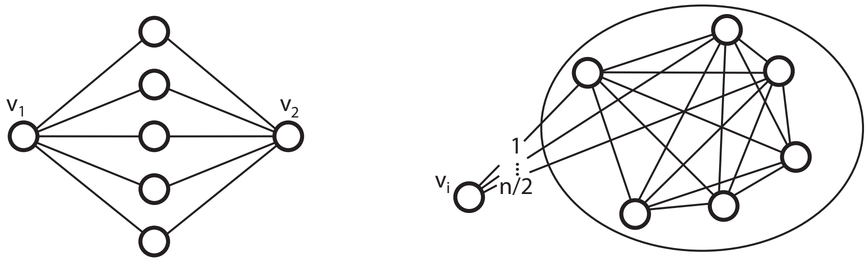

Figure 5.5: (a) Example where two CondrocetVertices v_1 and v_2 are not adjacent. (b) Worst case example for the expected payoff of CondorcetVertex v_i. The part of the graph in the circle is a complete graph.

5.2.1 Greedy Algorithm

Hansen et al. [10] provide a greedy algorithm for finding the (possibly emtpy) set of Condorcet points of a network in polynomial time. Our CondorcetVertices are not allowed lying on edges, therefore Hansen's algorithm is simplified to the pairwise comparison of vertices. The complexity is $O(n^3)$.

Algorithm Greedy. At the beginning all vertices are put in the set C_g of possible CondorcetVertices. For each possible pair (v_j, v_k) $v_j \in C_g, v_k \in V$ in turn the Distance Score $DS_k(j)$ is calculated. If $DS_k(j) > |V|/2$ then v_j is deleted from C_g. Stop if $C_g = \varnothing$, in this case the graph has no CondorcetVertex.

5.2.2 Tree

In the following we analyze our rumor game for the tree topology. Therefore we define a basic algorithm which finds an optimal node v_{opt} to publish the rumor for player 1. This node matches the $(1|1)$-centroid of the tree.

Algorithm TreeBasic. Let $T(V, E)$ be a tree. The player starts at a leaf $v_i \in V$. In each step the Distance Score $DS_j(i)$ of all neighbors $n_j \in N_i$ is computed. Player 1 moves to the neighbor with highest $DS_j(i)$ if she can improve herself and starts a new round. The algorithm stops at v_{opt} when no neighbor is better any more.

Lemma 5.2.4. *Given a tree $T(V, E)$ algorithm* TreeBasic *always finds a* CondorcetVertex.

Proof. In TreeBasic for each leaf v_i a unique path p_i is defined on which a player moves as long as she can improve her Distance Score $DS_j(i)$. The properties of a p_i are: (1) p_i is unique, (2) p_i does not contain any local maxima of DS_i. The uniqueness of p_i follows from the fact that each node has an unique parent node

and the tree does not contain any cycles. Property 2 follows from the definition of TreeBasic, where the player only moves to a neighboring node if she increases $DS_j(i)$. Therefore $DS_j(i)$ is monotonically increasing on p_i till v_{opt} is reached.
Lets assume player 2 finds a better node v'_{opt}, $DS'_{opt} > DS_{opt}$. Now consider player 1's v_{opt} as the new root of the tree $T'(V, E)$. The subtree T'_j which has neighbor n_j of v_{opt} as root contains v'_{opt}. The Distance Socre in subtree T'_j is maximized if $v'_{opt} = n_j$. However, for v_{opt} it holds that the Distance Score of each neighbor $DS_j(opt)$ is lower or equal than DS_{opt}. Therefore our assumption is wrong and player 2 can not find a better node. Moreover it holds $DS_j opt \leq |V|/2$ for every $v_j \in V$. Thus v_{opt} is a CondorcetVertex. □

5.2.3 D-dimensional Grid

Algorithm TreeBasic can be generalized to d-dimensional grids, compare algorithm *DGridBasic*.

Algorithm DGridBasic. In a d-dimensional grid $G_d(V, E)$ the player starts at an arbitrary node $v_i \in V$. In each round the Distance Score $DS_j(i)$ of all neighbors $n_j \in N_i$ is computed. The player moves to the best neighbor and starts a new round. The algorithm stops at v_{opt} where no neighbor is better any more.

Lemma 5.2.5. *In a d-dimensional grid algorithm* DGridBasic *always finds a* CondorcetVertex.

Proof. For any path p_i between a point v_i and the optimal solution v_{opt} following properties hold: (1) p_i does not contain any cycles. (2) There are no local maxima of $DS_j(i)$ on p_i. Player 1 moves on p_i to the neighbor with highest $DS_j(i)$ as long as she can improve her score according to algorithm DGridBasic. If no neighbor is better any more, then player 1 has found an optimal node.
Lets assume player 2 finds a better node. Because properties (1) and (2) hold this has to be a neighbor of the point found by player 1. However, the algorithm only terminates if it holds $DS_j \leq DS_i$ for all neighbors $v_j \in N$. Therefore our assumption was wrong and player 2 cannot find a better point than player 1. □

5.2.4 Small World Graphs

In this section we examine whether CondorcetVertices do exist in small world graphs. Therefore we look for CondorcetVertices in a Kleinberg small world graph with 64 nodes. If it holds for node v_j and any node $v_i \in V$ that $DS_i(j) \leq n/2$, then v_j is a CondorcetVertex. A simulation in a Kleinberg small world graph with 64 nodes finds in average 41.6 CondorcetVertices.

6 Conclusion

In this thesis we have presented the Rumor Game which models the spreading of information in networks. Two different models for two players were used to describe the problem. For one model we defined an extension to multiple players. For the propagation model two versions were specified, however only the Flooding Model, a basic version of the Threshold Model, was used in the analysis.
In the Basic Model we analyzed different underlying topologies, starting with star and line and then generalizing the examinations to d-dimensional grids. Moreover we examined this model for k players. The payoff for the players was determined for the corresponding topologies.
Furthermore simulations were performed in the Basic Model for different small world and random graphs on behalf of the tool Sinalgo. We were interested in rankings as the degree and the HITS value of nodes. In our simulations the highest degree node turned out to be the better to publish a rumor than the node with the highest HITS value.
The Bidding Model was as well analyzed for the different topologies star,line and d-dimensional grid. The payoff of each player and the existence of Nash equilibiria was determined.
In the section Location Theory we proved the np-hardness of the $(r|p)$-medianoid and $(1|p)$-centroid in our model and gave some analysis of the rumor game in trees and small world graphs.
Finally we thought about heuristics of the np-hard problem of finding the $(r|r)$-medianoid in graphs. Our approach was to subdivide the graph and to calculate in every part the $(1|1)$-medianoid.
During the research we encountered several questions and problems which exceed the scope of this thesis. We used only flooding as a propagation model, but defined

also its generalization the Threshold Model and the Independent Cascade Model. Analyzes with these models could give some further insights in the propagation of information.

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