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Distributed Computing

Network Creation Game on Blockchain Payment Channels

Semester Thesis

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Abstract

Payment channels emerged out of the scalability issues typically faced by cryptocurrencies. Together these payment channels can build a payment channels network - otherwise referred to as Layer 2 - on top of the blockchain. Possibly being part of the future of cryptocurrencies, the study of payment channels networks has become increasingly important. In this thesis, we study payment channels networks in a game theoretic setting.

We define a network creation game to model these networks, having players in the game pursue both betweenness and closeness centralities. With these incentives, players selfishly find their best strategy given a network configuration; inding the best strategy is NP-hard as we show. With our knowledge about the social optimum of our game and Nash equilibria attained in our parameters space, we bound the price of anarchy.

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Chapter 1

Introduction

With the proposal of Bitcoin in 2008 by Satoshi Nakamoto [1], cryptocurrencies started to emerge. While Bitcoin is still the largest cryptocurrency by market capitalization as of June 2019 [2], numerous more cryptocurrencies have surfaced over the past years [3, 4, 5]. Acceptance of cryptocurrencies is continuously rising, but Bitcoin's scalability is a bottleneck to its widespread adoption. Only 350'000 Bitcoin transactions per day were made on average in June 2019 [6] and Bitcoin is restricted to less than seven transactions per second with a block size limit of one megabyte [7]; significantly less than the daily average 150'000'000 transactions handled by Visa in 2012 [8] as well as the 65'000 transactions per second capacity of the VisaNet as of August 2017 [9].

Bitcoin's scalability barrier stems from its core, the distributed ledger, known as the blockchain. Every participant is required to keep a copy of the whole history of the network; allowing collective verification of transactions and circumnavigating intermediaries. Thus, the quantity of data on the blockchain is limited, creating competition to have one's transaction included on the blockchain [10]. This scalability issue on Bitcoin transactions calls for a new way of making transactions to keep the blockchain layer light.

Several approaches have been introduced to tackle this issue, the most prominent one being payment channels [7, 11, 12, 13]. Payment channels allow off-chain transactions through transaction replacement; reducing the use of the expensive and slow blockchain [10]. With the potential of payment channels to be the future of blockchain transactions, the study of their network topology becomes increasingly crucial.

In this thesis, we model payment channels networks as a network creation game. Network creation games have already been used to model the linking of autonomous systems on the internet to achieve a global connection [14] for instance and can generally be used to model networks created by independent players, as opposed to a central authority. In a network creation game, the incentive of a player is to minimize her cost by choosing to whom she connects.

We then use our model to study the topologies that emerge when players act selfishly. A graph is a Nash equilibrium when no player can decrease her cost by unilaterally changing her connections. We examine the existence and properties of these Nash equilibria for our model in this thesis. When possible, we bound the price of anarchy, the ratio of the social costs of the worst case Nash equilibrium and the social optimum [14] — allowing us to obtain an insight into the lack of coordination in payment channels networks when players act selfishly.

Contribution

We introduce a network creation game modeling the incentives of players in payment channels networks; combining betweenness and closeness centralities that have thus far only been studied independently in network creation games. First, we identify the social optimum for the entire parameter space of our game. Then we move on to analyze for prominent graphs, if and when they are a Nash equilibrium. Finally, with this combined knowledge upper bounds for the price of anarchy are determined.

Chapter 2

Background and Preliminaries

In this section we introduce the essential background and notation for our payment network creation game.

2.1 Network Creation Games

Typically, network creation games study Nash equilibria and the price of anarchy of networks formed by selfish players. Nash equilibria are game outcomes where no player has an incentive to deviate from her strategy. The worst case ratio between the social cost of a Nash equilibrium and the social cost of the social optimum is the price of anarchy, introduced by Koutsoupias et al. [15]. A low price of anarchy indicates that a selfishly acting player does not massively reduce network performance.

Players choose a strategy to minimize their cost in network creation games; selecting a subset of other players to whom to establish edges. Thus, the strategy combination creates an undirected graph.

2.2 Blockchain and Layer 2

Every node in the Bitcoin network stores the entire blockchain. The network converges to a unique state through the majority of nodes agreeing on the order of transactions in consequence of the addition to the blockchain.

Payment channels operate on top of the blockchain (Layer 2) and allow instantaneous off-chain transactions. Generally, a channel is set up by two parties following a defined protocol. The channel can then be used to make arbitrarily many transactions without committing each to the blockchain. When opening a channel, the parties pay a blockchain fee and place capital in the channel. The blockchain fee is the transaction fee to the miner; paid to have the transaction mined on a block and thereby published on the blockchain. Additionally, deposited capital funds future transactions on the channel and is not available for other on-chain transactions during the channel's lifetime. In our model, we assume a player single-handily initiates a channel to a subset of other players. Incoming channels are always accepted and once installed, the channels are undirected. While any player can typically choose the amount to lock, our model assumes fixed capital placed in channels. We make this simplification due to the complexity of the problem. The cost of opening a channel, including the capital, is set to one.

In addition to enabling parties connected by a payment channel to exchange funds off-chain, payment channels can also be used to route off-chain transactions between a sender and receiver pair not directly connected by a payment channel. Transactions between the sender and receiver can be routed through a path of channels, as long as there is enough capital on each edge. In the Lightning network [7] Hash Time Locked Contracts (HTLCs) are used for transaction routing. A HTLC requires the receiver to acknowledge the receipt of the payment within a given time frame by providing cryptographic proof. Otherwise, her ability to claim the payment forfeits. Thus, together the payment channels build a payment channels network. In the network players receive a payment when transactions are routed through them. This payment is a transaction fee. We define a payment channels network as an undirected graph consisting of a set of players$V$, the nodes, and a set of payment channels E, the edges. For the sake of simplicity, we assume a fixed transaction fee.

The payments received by a player for providing gateway services to other players' transactions are modeled by her betweenness centrality. Betweenness centrality was first introduced as a measure of a players importance in a social networks by Freeman et al. [16]. A players betweenness centrality in a graph $G(V, E)$ is given by

$$\sum_{\substack{s,r \in V: \\ s \neq r \neq u, m(s,r)>0}} \frac{m_u(s,r)}{m(s,r)},$$

where $m_u(s, r)$ is the number of shortest paths between sender s and receiver r that route through player u and $m(s, r)$ is the total number of shortest paths between s and r. Intuitively, betweenness centrality of player u is a measure of the expected number of sender and receiver pairs that would choose to route their transactions through her in a payment channels network. Providing an insight into the transaction fees a player is expected to receive, the betweenness centrality lends itself to reflect the motivation of a player in a payment channel network to maximize the payments secured through providing transaction gateway services.

Furthermore, we model the fees encountered by a player when having her transactions routed through the network through her closeness centrality. Closeness centrality measures the sum of distances to all other players. With the transaction fees fixed per hop in our model, the distance to a player r estimates the costs encountered by player u when sending a transaction to player r. Therefore, the sum of distances to all other players is a natural proxy for the fees u faces for making transactions when assuming uniform transactions; a simplifying assumption we make in our model.

Thus, the combination of betweenness and closeness centralities accurately encapsulates the incentives inherent to players in a payment channels network.

Chapter 3

Model

A payment channels network game consists of n players $V = \{0, 1, \ldots, n-1\}$. Each player u can initiate channels to a set of players and automatically places a fixed capital in those channels. The total cost of establishing a channel is one, representing the blockchain fee and capital placed in the channel. It is assumed that incoming edges are always accepted. The strategy of player u is denoted by s_u, and the set $S_u = 2^{[n]-\{u\}}$ defines u's strategy space. We consider the graph $G[s]$. $G[s]$ is the underlying undirected graph of $G_0[s] = \left([n], \bigcup_{u\in[n]}\{u\} \times s_u\right)$, where $s = (s_0, \ldots, s_{n-1}) \in S_0 \times \cdots \times S_{n-1}$ is a strategy combination. While a channel can possibly be created by both endpoints, this will never be the case in a Nash equilibrium.

The cost of player u under policy s is

$$\text{cost}_u(s) = |s_u| + b \cdot \text{betweenness}_u(s) + c \cdot \text{closeness}_u(s),$$

where $b \geq 0$ is the betweenness weight and $c > 0$ the closeness weight.

The betweenness of u is measured as follows:

$$\text{betweenness}_u(s) = (n-1)(n-2) - \sum_{\substack{s,r\in[n]:\\ s\neq r\neq u, m(s,r)>0}} \frac{m_u(s,r)}{m(s,r)}.$$

We subtract u's betweenness centrality as defined by Freeman et al. [16] from her maximum possible betweenness centrality to ensure that the social cost is always positive - avoiding cases where price of anarchy is undefined.

On the other hand, the closeness of u is measured as follows:

$$\text{closeness}_u(s) = \sum_{r\in[n]-u} d_{G[s]}(u,r) - 1,$$

where $d_{G[s]}(u,r)$ is the distance between u and r in the graph $G[s]$ and $d_{G[s]}(u,r)-1$ represents the number of transaction fees encountered by u when routing a transaction to r through the network. Letting $c > 0$ ensures that the graph is always connected, as a player's cost is infinite in a disconnected graph.

The objective of player u is $\min_{s_u} \text{cost}_u(s)$, and the social cost is the sum off all players' costs: $\text{cost}(s) = \sum_{u \in [n]} \text{cost}_u(s)$. Thus, the social optimum is $\min_s \text{cost}(s)$.

Chapter 4

Payment Channels Network Creation Game

To gain an insight into the emerging topologies and their lack of coordination when players' act egocentrically, we will first analyze the social optimum for our model of payment channels networks. After studying if and when prominent graphs are Nash equilibria, we finish by bounding the price of anarchy.

4.1 Social Optimum

By definition of the cost function the social cost is

$$\begin{aligned}\text{cost}(s) = &\sum_{u\in[n]} \text{cost}_u(s)\\ =&|E(G)| + b\cdot \sum_{u\in[n]} \text{betweenness}_u(s) + c\cdot \sum_{u\in[n]} \text{closeness}_u(s),\end{aligned}$$

for any graph where no channel is paid by both endpoints. This constraint is met for all Nash equilibria. To lower bound the social cost, we will first simplify the social cost expression.

Lemma 4.1 (Theorem 1 [17])**.** *The average betweenness $\overline{B}(G)$ in a connected graph G can be expressed as: $\overline{B}(G) = (n-1)(\bar{l}(G)-1)$, where $\bar{l}(G)$ is the average distance in G.*

Lemma 4.1 is proven in [17] and relates the average betweenness and distance in a connected graph. We will show how to express the social cost directly in terms of the number of edges and the sum of the players' closeness centrality costs; facilitating further analysis.

Lemma 4.2. *The social cost in G is given by*

$$\textit{cost}(s) = |E(G)| + b\cdot n\cdot (n-1)(n-2) + (c-b)\cdot \sum_{u\in[n]} \textit{closeness}_u(s).$$

Proof. According to Lemma 4.1 the social cost can be expressed as

$$\begin{aligned}
\text{cost}(s) =& |E(G)| + b \cdot \sum_{u\in[n]} \text{betweenness}(u) + c \cdot \sum_{u\in[n]} \text{closeness}(u) \\
=& |E(G)| + b \cdot \sum_{u\in[n]} \left((n-1)(n-2) - \sum_{\substack{s,r\in[n]:\\ s\neq r\neq u, m(s,r)>0}} \frac{m_u(s,r)}{m(s,r)} \right) \\
&+ c \cdot \sum_{u\in[n]} \sum_{r\in[n]-u} \big(d_{G[s]}(u,r) - 1\big) \\
=& |E(G)| + b \cdot n \cdot (n-1)(n-2) - b \cdot n \cdot \overline{B}(G) + c \cdot n \cdot (n-1)(\bar{l}(G) - 1) \\
=& |E(G)| + b \cdot n \cdot (n-1)(n-2) + (c-b) \cdot n \cdot (n-1)(\bar{l}(G) - 1) \\
=& |E(G)| + b \cdot n \cdot (n-1)(n-2) + (c-b) \cdot \sum_{u\in[n]} \sum_{r\in[n]-u} \big(d_{G[s]}(u,r) - 1\big)
\end{aligned}$$

for all $b \geq 0$ and $c > 0$. □

Lemma 4.3 provides bounds for the distance of a graph G,

$$d(G) = \frac{1}{2} \sum_{u\in[n]} \sum_{r\in[n]-u} (d_G(u,r) - 1)\,;$$

useful for finding the social optimum for our game.

Lemma 4.3 (Theorem 2.3 [18]). *If G is a connected graph with n vertices and k edges then $n \cdot (n-1) \leq d(G) + k \leq \frac{1}{6} \cdot \big(n^3 - 5 \cdot n - 6\big)$.*

In [18] Lemma 4.3 is proven and the path graph achieves the upper bound. This can be used to find the social optimum. Dependent on the weights b and c, the social optimum for the payment channels network creation game is given in Theorem 4.4 and Figure 4.1 illustrates this parameter space.

Theorem 4.4. *The social optimum is a complete graph for $c > \frac{1}{2} + b$, a star graph for $b \leq c \leq \frac{1}{2} + b$ and a path graph for $c < b$.*

Proof. Using Lemma 4.2 we can lower bound the social cost for $c \geq b$ as follows:

$$\begin{aligned}
\text{cost}(s) =& |E(G)| + b \cdot n \cdot (n-1)(n-2) + \underbrace{(c-b)}_{\geq 0} \cdot \sum_{u\in[n]} \sum_{r\in[n]-u} \big(d_{G[s]}(u,r) - 1\big) \\
\geq& |E(G)| + b \cdot n \cdot (n-1)(n-2) + (c-b)(n \cdot (n-1) - 2|E|) \\
=& (1 - 2 \cdot (c-b)) \cdot |E(G)| + b \cdot n \cdot (n-1)(n-2) + (c-b)(n \cdot (n-1))
\end{aligned}$$

since every pair of nodes that is not connected by an edge is at least distance two apart [14]. This lower bound is achieved by any graph with diameter at most two.

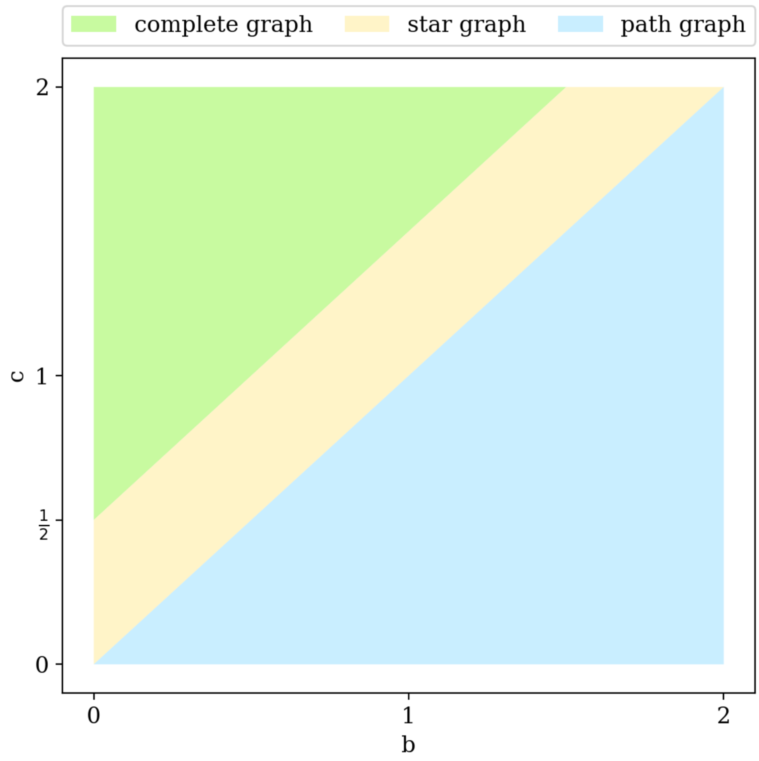


Figure 4.1: Parameter map for social optimum of game.

It follows that for $c > \frac{1}{2} + b$ the social optimum is a complete graph, maximizing $|E|$, and for $b \leq c \leq \frac{1}{2} + b$ the social optimum is a star, minimizing $|E|$.

To find the social optimum for $c < b$, we rewrite the social cost as

$$\begin{aligned}\text{cost}(s) =& |E(G)| + b \cdot n \cdot (n-1)(n-2) - (b-c) \cdot \sum_{u \in [n]} \sum_{r \in [n]-u} \left(d_{G[s]}(u,r) - 1\right) \\ =& |E(G)| - 2 \cdot (b-c) \cdot d(G) + b \cdot n \cdot (n-1)(n-2) + (b-c) \cdot n \cdot (n-1),\end{aligned}$$

For a connected graph the social cost is then minimized for a tree, as

$$|E(H)| - a \cdot d(H) > |E(G)| - a \cdot d(G)$$

if G is a supergraph of H and $a > 0$.

Using Lemma 4.3, we get that

$$\begin{aligned}\text{cost}(s) =& |E| + b \cdot n \cdot (n-1)(n-2) - (b-c) \cdot \sum_{u \in [n]} \sum_{r \in [n]-u} \left(d_{G[s]}(u,r) - 1\right) \\ \geq& \left(1 + b \cdot n \cdot (n-2) + \frac{b-c}{3} n \cdot (n-2)\right)(n-1)\end{aligned}$$

is a lower bound for the social cost and this lower bound is achieved by a path graph. □

4.2 Nash Equilibria

A naive approach to finding Nash equilibria is to start with a graph and continuously compute a player's best response in the game. Adjusting the strategy according to the best response until a Nash equilibrium is reached. However, Theorem 4.5 shows that it is NP-hard to calculate a player's best response.

Theorem 4.5. *Given a strategy $s \in S_0 \times \cdots \times S_{n-1}$ and $u \in [n]$, it is NP-hard to computed the best response of u.*

Proof. The following proof is adapted from Proposition 1 in [14].

Given the configuration of the rest of the graph, player u has to compute her best response; a subset of players to build channels to such that her cost is minimized. For $b = 0$ and $0.5 < c < 1$ and no incoming links from the rest of the graph, we know that the diameter of G can be at most 2. Additionally, making more than the minimum number of required links, only improves the distance term by c, which is strictly smaller than the cost of establishing a link. Thus, u's strategy is a dominating set for the rest of the graph.

The cost of u is minimized when the size of the subset is minimized. The minimum size dominating set corresponds to u's best response. Hence, it is NP-hard to compute a player's best response by reduction from the dominating set. □

Therefore, with this in mind, we analyze prominent graph topologies theoretically, to see if and when they are Nash equilibria in our game. However, complementary to the theoretical analysis we also create a simulation to get insights into emerging graph topologies for a small number of players.

4.2.1 Complete Graph

For large c the complete graph is the only Nash equilibrium as stated in Theorem 4.6.

Theorem 4.6. *For $c > 1$, the only Nash equilibrium is the complete graph.*

Proof. The addition of an edge by a player never increases her betweenness cost. Thus, by the definition of the cost function any Nash equilibrium cannot be missing any edges whose addition would reduce a players closeness by more than one, the cost of building an edge. As $c > 1$, no edge can be missing in the graph and the only Nash equilibrium is the complete graph. □

Additionally, the complete graph is also a Nash equilibrium for $c = 1$, but it is not necessarily the only one. Theorem 4.7 on the other hand shows when the complete graph is not a Nash equilibrium.

Theorem 4.7. *For $c < 1$ and $n \geq 3$, the complete graph is never a Nash equilibrium.*

Proof. In a complete graph the removal of an edge by a player does not change her betweenness cost and her closeness cost is increased by c. Thus, the cost of a player would decrease when removing one edge. Therefore, the complete graph is not a Nash equilibrium for $c < 1$. □

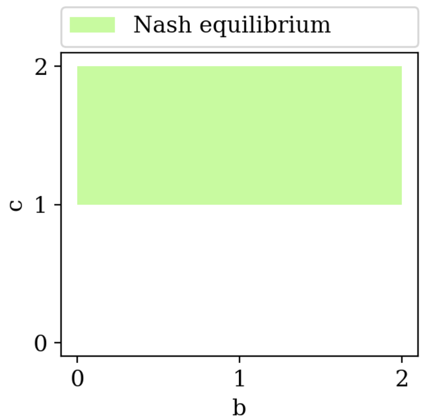


Figure 4.2: Parameter map for complete graph.

These results are combined in Figure 4.2. Figure 4.2 visualizes when the complete graph is a Nash equilibrium in our game.

4.2.2 Circle Graph

For small values of n, the circle graph can be a Nash equilibrium depending on the weights b and c. The circle graph and the complete graph are the same for $n = 3$. Thus, for $n = 3$ the circle graph is a Nash equilibrium if and only if $c \geq 1$.

Proposition 4.8. *For $n = 4$, the circle graph is a Nash equilibrium if and only if $c \leq 1 \leq b + 2 \cdot c$.*

Proof. Adding a link to the only player one is not directly connected to, does not decrease a player's betweenness cost. It is not beneficial for a player to initiate an additional link, if the closeness cost reduction is not bigger than the link cost of one. Thus, a player does not add an additional edge for

$$c \leq 1.$$

A player's change in cost when removing a single link is given by

$$\Delta\text{cost}_u(\text{remove 1 link}) = -1 + b + 2 \cdot c.$$

Hence, a player cannot reduce her cost through the removal of a single link if $1 \leq b + 2 \cdot c$.

Additionally, a player with two outgoing links will never eliminate both without adding another link. Her cost would be infinite otherwise. Exchanging a single link with a new link to the player one was not previously connected to, never yields a negative change in cost and is therefore never a player's best response.

Finally, the change in cost when removing two links and adding a new link to the player one was not previously connected to is given by

$$\Delta\text{cost}_u(\text{remove 2 \& add 1 link}) = -1 + b + c,$$

but this bound is more restrictive than the previous one, and there is no need for a player to have more than one outgoing edge.

Thus, the circle graph with $n = 4$ is a Nash equilibrium for $c \leq 1 \leq b+2\cdot c$. □

Proposition 4.9. *For $n = 5$, the circle graph is a Nash equilibrium if and only if $b + c \leq 1 \leq 2 \cdot b + 4 \cdot c$.*

Proof. The change in cost for the addition of links to the players, a player was not directly connected to previously, is given by

$$\Delta\text{cost}_u(\text{add } m \text{ links}) = m - m \cdot b - m \cdot c,$$

where $m \in \{1, 2\}$. Thus, a player can reduce her cost by adding more links when $1 \leq b + c$.

If a player in the circle graph removes one outgoing link the change in cost is

$$\Delta\text{cost}_u(\text{remove 1 link}) = -1 + 2 \cdot b + 4 \cdot c.$$

A player with an outgoing link benefits from the removal if $1 \geq 2 \cdot b + 4 \cdot c$. On the other hand, a player with two outgoing edges will never remove both links without adding a new link as the graph would become disconnected otherwise. Additionally, she never benefits more from exchanging links as the change in cost is non-negative. When replacing both her links by a new link, the change in cost is

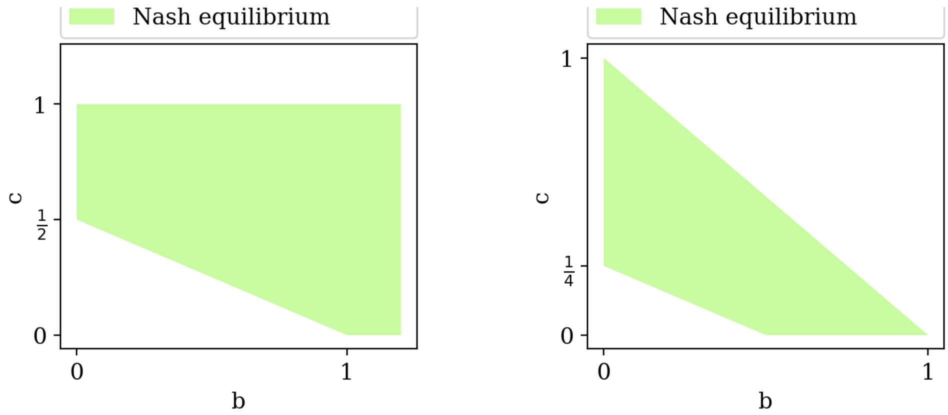
$$\Delta\text{cost}_u(\text{remove 2 \& add 1 link}) = -1 + 2 \cdot b + 2 \cdot c.$$

However, this leads to a more restrictive bound than just removing one link and no player in a circle graph needs more than one outgoing link.

We have shown that for $n = 5$ the circle graph is a Nash equilibrium if and only if

$$b + c \leq 1 \leq 2 \cdot b + 4 \cdot c.$$

□



(a) $n = 4$ (b) $n = 5$

Figure 4.3: Parameter map for circle graph.

Propositions 4.8 and 4.9 show that for small n, the circle graph can be a Nash equilibrium depending on the weights. The parameter space of the circle graph for $n = 4$ and $n = 5$ is shown in Figure 4.3a and Figure 4.3b, respectively. However, for large n Theorem 4.10 states that the circle graph is never a Nash equilibrium.

Theorem 4.10. *There exists a $N > 0$, such that for all $n \geq N$ the circle graph is never a Nash equilibrium.*

Proof. We will show that any player with one outgoing edge in a circle graph with $n \geq N$ players, has an incentive to change strategy. Thus, the circle graph cannot be a Nash equilibrium.

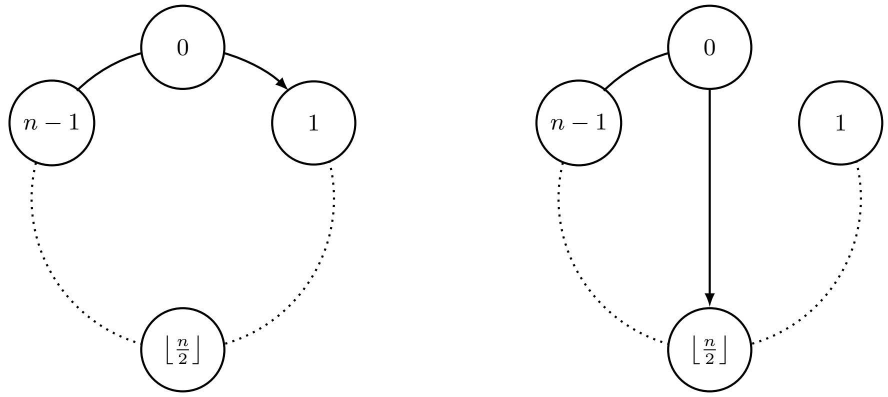
Consider the circle graph in Figure 4.4a. Without loss of generality, assume that player 0 has one outgoing edge to player 1. As the equations for 0's betweenness and closeness differ for n even or odd, we will use asymptotic notation throughout the following analysis.

In the circle graph, strategy s, the betweenness of 0 is

$$\text{betweenness}_0(s) = \frac{3}{4} \cdot n^2 + o\left(n^2\right)$$

and the 0's closeness is

$$\text{closeness}_0(s) = \frac{1}{4} \cdot n^2 + o\left(n^2\right).$$



(a) Strategy s (b) Strategy $\tilde{s}$

Figure 4.4: Strategy change of player 0.

Now, player 0 removes the link to player 1 and initiates a new link to player $\lfloor \frac{n}{2} \rfloor$, seen in Figure 4.4b. We will refer to this strategy as $\tilde{s}$. The first part of 0's betweenness cost reduction comes from the shortest paths of players in the 1^{st} and 2^{nd} quadrant to the 4^{th} quadrant, as well as the other way around; the quadrants are as shown in Figure 4.5. These shortest paths go through the shortcut and subtract

$$2 \cdot \left(\frac{n}{2} + o(n)\right) \cdot \left(\frac{n}{4} + o(n)\right) = \frac{n^2}{4} + o\left(n^2\right),$$

from 0's betweenness cost. The second part stems from nodes in the 1^{st} and 3^{rd} quadrant using node 0 as a gateway in the cycle. We have a further betweenness cost reduction of

$$\frac{1}{4} \cdot \left(\frac{n}{2}\right)^2 + o\left(n^2\right) = \frac{n^2}{16} + o\left(n^2\right).$$

Thus, the betweenness of 0 with strategy $\tilde{s}$ is at most

$$\text{betweenness}_0(\tilde{s}) = n^2 - \frac{n^2}{4} - \frac{n^2}{16} + o\left(n^2\right) = \frac{11}{16} \cdot n^2 + o\left(n^2\right).$$

The closeness of player 0 to players in the 3^{rd} and 4^{th} quadrant is

$$\frac{1}{4} \cdot \left(\frac{n}{2}\right)^2 + o\left(n^2\right) = \frac{n^2}{16} + o\left(n^2\right),$$

and to players in the 1^{st} and 2^{nd} quadrant 0's closeness is

$$\frac{1}{2} \cdot \left(\frac{n}{2}\right)^2 + o\left(n^2\right) = \frac{n^2}{8} + o\left(n^2\right).$$

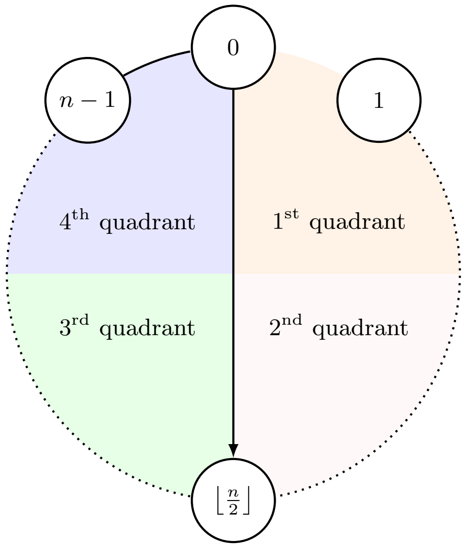


Figure 4.5: Quadrants of graph.

Therefore, 0's closeness is

$$\text{closeness}_0(\tilde{s}) = \frac{n^2}{16} + \frac{n^2}{8} + o\left(n^2\right) = \frac{3}{16} \cdot n^2 + o\left(n^2\right).$$

Player 0's change in cost is

$$\begin{aligned}\Delta\text{cost}_u(s \text{ to } \tilde{s}) &= \left(\frac{11}{16}n^2 - \frac{3}{4}n^2 + o\left(n^2\right)\right) \cdot b + \left(\frac{3}{16}n^2 - \frac{1}{4}n^2 + o\left(n^2\right)\right) \cdot c \\ &= -\left(\frac{1}{16}n^2 + o\left(n^2\right)\right)(b+c).\end{aligned}$$

As player 0 would choose strategy $\tilde{s}$ over strategy s for $\Delta\text{cost}_u(s \text{ to } \tilde{s}) < 0$, there exists a $N > 0$, such that for $n \geq N$ player the circle graph is never a Nash equilibrium. □

We note that simulations suggest that for $n \geq 6$ the circle graph is never a Nash equilibrium. Parameter sweeps indicating that $N = 6$ can be found in Appendix A.2.

4.2.3 Star Graph

In a star graph the player in the center has minimal closeness and betweenness costs; all other players have maximal betweenness cost. While this does not directly appear to be a stable network, Theorem 4.11 suggest that the star graph is a Nash equilibrium for smaller values of b and c. These results are depicted in Figure 4.6.

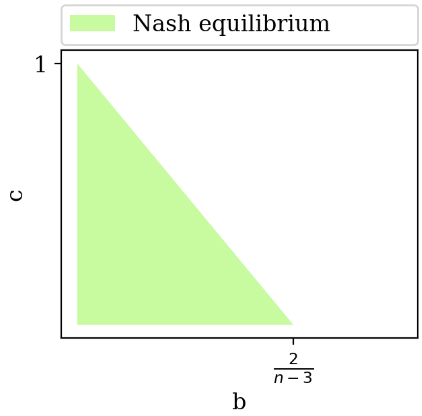


Figure 4.6: Parameter map for star graph.

Theorem 4.11. *For $n \geq 3$, the star is always a Nash equilibrium if and only if $0 \leq 1 - \frac{n-3}{2}b - c$.*

Proof. To show that the star is always a Nash equilibrium for $n \geq 3$ and $0 \leq 1 - \frac{n-3}{2}b - c$, we will consider a star graph consisting of n players $V = \{0, 1, \ldots, n-1\}$. Without loss of generality we assume that player 0 is the center of the star.

No player in the star graph has an incentive to remove an edge, as this would lead to infinite cost. Thus, player 0 has no incentive to change strategy, as she is connected to everyone.

In the following we will consider star graph where all links are initiated by player 0 and star graphs where at least one link is initiated by another player separately.

If all links are initiated by player 0, players 1, 2, ..., $n-1$ are all in an equivalent position and it is therefore sufficient to solely consider player 1. Player 1 would only add links, if this leads to a decrease in her cost. Initiating an edge to player 0 would only increase her cost. Additionally, for the remaining $n-2$ players, it only matters to how many player 1 connects to. The change in cost when adding m, where $1 \leq m \leq n-2$, edges is given by

$$\Delta \text{cost}_1(\text{add } m \text{ links}) = m - \frac{m \cdot (m-1)}{2} b - m \cdot c.$$

Thus, player 1 will change strategy if $\Delta \text{cost}_1(\text{add } m \text{ links}) < 0$. The change in cost is minimized for $m = n-2$.

In star graphs where at least one player other than 0 initiates a link, players that have not outgoing links are in the same position as those analyzed previously. Thus, it suffices to consider player i, where $i \neq 0$, that has one outgoing link. In addition to only initiating new links, player i can remove the link to player 0 and initiates l, where $1 \leq l \leq n-2$, new links. The change in cost is then given as

$$\Delta \text{cost}_i(\text{add } l \text{ links}) = (l-1) - \frac{l \cdot (l-1)}{2} b - (l-1) \cdot c.$$

However, this leads to more restrictive bounds and there is no need for players other than player 0 to have outgoing links.

Thus, the star is a Nash equilibrium if and only if

$$0 \leq 1 - \frac{n-3}{2} b - c.$$

□

4.2.4 Complete Bipartite Graph

The star graph, analyzed in Section 4.2.3 is a complete bipartite graph where one group has size one. In this section we will analyze more general complete bipartite graphs or bicliques $K_{r,s}$, where r is the size of the smaller and s is the size of the bigger subset. Every node from one subset is connected to all nodes from the other subset in a complete bipartite graph.

Theorem 4.12. *The complete bipartite graph $K_{r,s}$ with $3 \leq r \leq s$ is stable if and only if $\frac{s-2}{r+1} b + c + \leq 1 \leq \min \left\{ \frac{s}{r} b + \frac{s+r-3}{s-1} c, \min\{\alpha, \beta\} \cdot b + c \right\}$, where $\alpha = \frac{s \cdot (s-1)}{r \cdot (s-2)}$ and $\beta = \frac{1}{s-r+1} \left(\frac{s \cdot (s-1)}{r} - \frac{(r-2)(r-1)}{s+1} \right)$.*

Proof. Additional links can only be created within a subset in a complete bipartite graph. Similarly, to adding links in a star graph the change in cost when adding m links is given by

$$\Delta \text{cost}_u(\text{add } m \text{ links}) = m - \frac{m \cdot (m-1)}{l+1} b - m \cdot c = 1 - \frac{m-1}{l+1} b - c,$$

where $l \in \{r, s\}$ is the size of the subset not including the player.

A player changes strategy, when $\Delta \text{cost}_u(\text{add } m \text{ links}) < 0$. The change in cost is minimized, when m is maximized and $l = r$. m can therefore be $s-1$ at most. Thus, the upper bound for $K_{r,s}$ being a Nash equilibrium is

$$1 \geq \frac{s-2}{r+1} b + c.$$

Players in the subset of size r, benefit more from a link to the other subset, as their betweenness cost is smaller. Hence, to find a lower bound for b and c we only consider complete bipartite graphs, in which all links are established from the smaller subset, as seen in Figure 4.7a. Without loss of generality we will only consider player u in the following analysis. It is not reasonable for player u to remove all her links without adding any new links, as her cost would become infinite. Depending on the other parameters, it might be more optimal to remove all her previous links and only connect to one player in her subset (Figure 4.7b), connect to one player in her subset and one player from the other subset (Figure 4.7c), or to remove all her previous links and instead connect to all other players in her subset (Figure 4.7d).

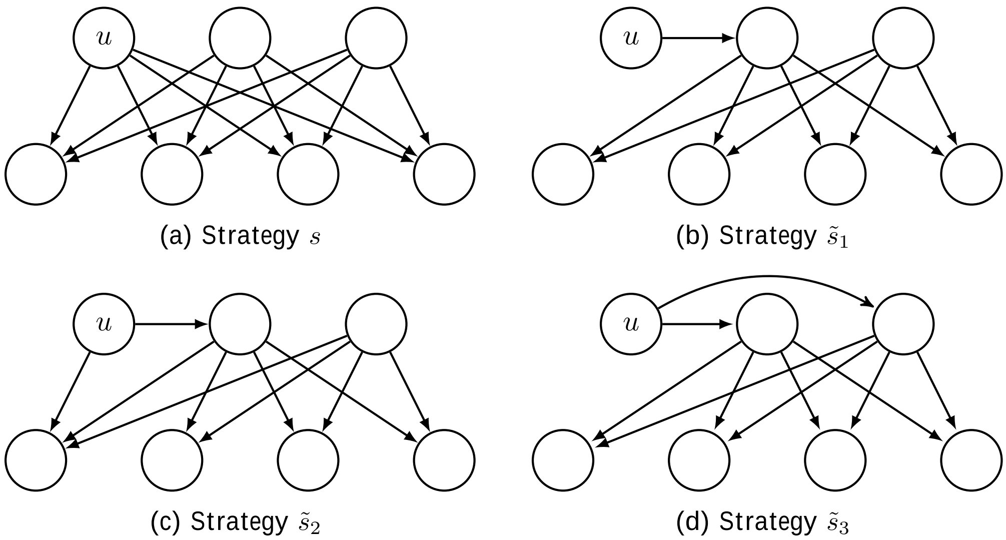


Figure 4.7: Strategy deviations of player u.

When player u changes to strategy $\tilde{s}_1$, seen in Figure 4.7b the change in cost is as follows:

$$\Delta\text{cost}_u(s \text{ to } \tilde{s}_1) = -(s-1) + \frac{s \cdot (s-1)}{r} b + (s+r-3) \cdot c$$

as player u initiates $s-1$ less links then before - loosing all her previous betweenness. Additionally, she is one hop further away from all other players except for the one she connects to directly. It follows, that for the above strategy to be less preferable than the complete bipartite graph for player u, if

$$1 \leq \frac{s}{r} b + \frac{s+r-3}{s-1} c.$$

Player u's change to strategy $\tilde{s}_2$ (Figure 4.7c) leads to $s-2$ less links initiated by her. The player is further away from $s-1$ players from the other subset and closer to one in her own. All transaction routing potential is lost. Therefore, the change in cost is given by

$$\Delta\text{cost}_u(s \text{ to } \tilde{s}_2) = 2 - s + \left(\frac{s \cdot (s-1)}{r}\right) b + (s-2) \cdot c.$$

Hence, for this strategy to be less preferable then the complete bipartite graph,

$$1 \leq \left(\frac{s \cdot (s-1)}{r \cdot (s-2)}\right) b + c = \alpha \cdot b + c.$$

When severing all previous links and connecting to all players in her subset instead, strategy $\tilde{s}_3$ (Figure 4.7d), player u builds $s - r + 1$ less links then before. Furthermore, she is closer to players previously in her own subset and further away from the rest. While player u can now transmit transactions of players previously in her own subset, she is no longer a preferable intermediary for players previously in the other subset. Therefore, the change in cost is given by

$$\Delta \text{cost}_u(s \text{ to } \tilde{s}_3) = r - s + 1 + \left(\frac{s \cdot (s-1)}{r} - \frac{(r-1)(r-2)}{s+1}\right) b + (s - r + 1) \cdot c.$$

Hence, for this strategy to be less preferable then the complete bipartite graph for player u,

$$1 \leq \frac{1}{(s-r+1)} \left(\frac{s \cdot (s-1)}{r} - \frac{(r-1)(r-2)}{s+1}\right) b + c = \beta \cdot b + c.$$

To summarize, the complete bipartite graph $K_{r,s}$ is a Nash equilibrium for

$$\frac{s-2}{r+1} b + c \leq 1 \leq \min\left\{\frac{s}{r} b + \frac{s+r-3}{s-1} c, \min\{\alpha, \beta\} \cdot b + c\right\}.$$

□

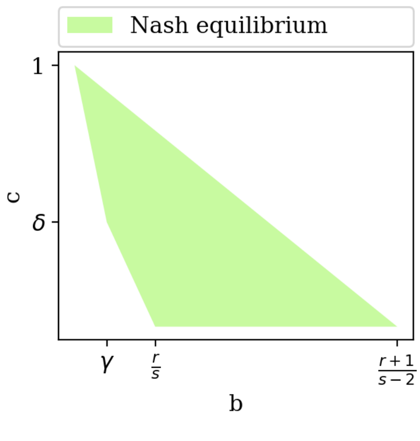


Figure 4.8: Parameter map for complete bipartite graph $K_{r,s}$.

The parameter map for the complete bipartite graph is drawn in Figure 4.8. There (γ, δ) is the intersection between $1 = \frac{s}{r} b + \frac{s+r-3}{s-1} c$ and $1 = \min\{\alpha, \beta\} \cdot b + c$.

4.2.5 Simulation

To better understand the behaviour of a player in our network creation game, we implement a simulation of the game [19]. Our simulation enumerates all Nash equilibria for a given number of players n, as well as the weights for the betweenness and closeness costs. However, this is only feasible for small n. Parameter sweeps for the weights b and c can also be performed to see when a given topology is a Nash equilibrium. Some parameter sweeps for topologies previously analyzed can be found in Appendix A. Finally, starting from an initial graph the progression of the game can be simulated.

4.3 Price of Anarchy

The price of anarchy provides an insight to the lack of coordination, when players act selfishly. Where the price of anarchy is low, selfish actors do not heavily degrade network performance. However, a high price of anarchy indicates that network formation by a central authority would significantly increase performance.

For $c > 1$ we can determine the price of anarchy exactly, as we have found the social optimum for our entire parameter space and are aware of all Nash equilibria for $c > 1$.

Corollary 4.13. *For $c > 1$ and $c > \frac{1}{2} + b$, the price of anarchy is $\rho(G) = 1$.*

Proof. The only Nash equilibrium for $c > 1$ is the complete graph as stated by Theorem 4.6. As the social optimum for $c > \frac{1}{2} + b$ is also the complete graph (Theorem 4.4), the price of anarchy is

$$\rho(G) = 1,$$

for $c > 1$ and $c > \frac{1}{2} + b$. □

Corollary 4.14. *For $c > 1$ and $b \leq c \leq \frac{1}{2} + b$, the price of anarchy is*

$$\rho(G) = \frac{\left(\frac{1}{2} + (n-2) \cdot b\right) \cdot n}{1 + (c + b \cdot (n-1))(n-2)}.$$

Proof. For $c > 1$ and $b \leq c \leq \frac{1}{2} + b$ the only Nash equilibrium is the complete graph (Theorem 4.6) and according to Theorem 4.4, the social optimum is the star graph. Thus, the price of anarchy is given by

$$\begin{aligned}\rho(G) =& \frac{\text{cost(complete graph)}}{\text{cost(star graph)}} \\ =& \frac{\left(\frac{1}{2} + (n-2) \cdot b\right)(n-1) \cdot n}{(1 - 2(c-b) + (c-b) \cdot n + b \cdot (n-2) \cdot n)(n-1)} \\ =& \frac{(\left(\frac{1}{2} + (n-2) \cdot b\right) \cdot n}{1 + (c + b \cdot (n-1))(n-2)}.\end{aligned}$$

□

Corollary 4.15. *For* $1 < c < b$ *, the price of anarchy is*

$$\rho(G) = \frac{\left(\frac{1}{2} + (n-2) \cdot b\right) \cdot n}{1 + \left(\frac{2}{3}b - \frac{1}{3}c\right) \cdot n \cdot (n-2)}.$$

Proof. For $1 < c < b$ the only Nash equilibrium is the complete graph (Theorem 4.6) and the social optimum is a path graph (Theorem 4.4). The price of anarchy is given by

$$\begin{aligned}\rho(G) =& \frac{\text{cost(complete graph)}}{\text{cost(path graph)}} \\ =& \frac{\left(\frac{1}{2} + (n-2) \cdot b\right)(n-1) \cdot n}{\left(1 + b \cdot n \cdot (n-2) - \frac{b-c}{3} \cdot n \cdot (n-2)\right)(n-1)} \\ =& \frac{(\left(\frac{1}{2} + (n-2) \cdot b\right) \cdot n}{1 + \left(\frac{2}{3}b - \frac{1}{3}c\right) \cdot n \cdot (n-2)}.\end{aligned}$$

□

Combining the results of Corollary 4.13, 4.14 and 4.15 allows us to upper bound the price of anarchy to a constant for $c > 1$, as stated in in Corollary 4.16. This upper bound is also a tight bound, as the price of anarchy is always greater or equal to one by definition.

Corollary 4.16. *For* $c > 1$*, the price of anarchy* $\rho(G) = \mathcal{O}(1)$*.*

Proof. For $c > 1$ and $c > \frac{1}{2} + b$, the price of anarchy is one and therefore it is also $\mathcal{O}(1)$.

We have that for $c > 1$ and $b \leq c \leq \frac{1}{2} + b$,

$$\rho(G) = \frac{\left(\frac{1}{2} + (n-2) \cdot b\right) \cdot n}{1 + (c + b \cdot (n-1))(n-2)} = \mathcal{O}\left(\frac{b \cdot n^2}{b \cdot n^2}\right) = \mathcal{O}(1),$$

and for $1 < c < b$,

$$\rho(G) = \frac{\left(\frac{1}{2} + (n-2) \cdot b\right) \cdot n}{1 + \left(\frac{2}{3}b - \frac{1}{3}c\right) \cdot n \cdot (n-2)} = \mathcal{O}\left(\frac{b \cdot n^2}{b \cdot n^2}\right) = \mathcal{O}(1).$$

Thus, for $c > 1$ we have $\rho(G) = \mathcal{O}(1)$.

□

For small b and c we can also upper bound the price of anarchy as follows:

Theorem 4.17. *For $c + b < \frac{1}{n^2}$, the price of anarchy is $\mathcal{O}(1)$.*

Proof. For $c+b < \frac{1}{n^2}$, all Nash equilibria are trees. Unless the distance to a player is infinite, no player in the network will have an incentive to build an edge.

As both the maximum possible change in $\text{betweenness}_u(s)$ and $\text{closeness}_u(s)$ for a node u in a connected graph is less than n^2 and all Nash equilibria are connected,

$$\Delta\text{cost}_u(s) > -n^2 \cdot c - n^2 \cdot b + 1.$$

We require $\Delta\text{cost}_u(s) \geq 0$ such that u does not benefit from initiating an additional channel. Thus, for $c + b \leq \frac{1}{n^2}$ all Nash equilibria are spanning trees.

For $c + b \leq \frac{1}{n^2}$ the social optimum is also a spanning tree, as it is either the star or path graph. It easily follows that for $c + b \leq \frac{1}{n^2}$ and all spanning trees $\text{cost}(s) = \Theta(n)$ and therefore the price of anarchy is $\mathcal{O}(1)$. □

Finally, for $c + b \geq \frac{1}{n^2}$ and $c < 1$ we find an upper bound of $\mathcal{O}(n)$ for the price of anarchy.

Theorem 4.18. *For $c + b \geq \frac{1}{n^2}$ and $c < 1$, the price of anarchy is $\mathcal{O}(n)$.*

Proof. The price of anarchy is

$$\rho(G) = \mathcal{O}\left(\frac{|E(G)| + n^3 \cdot b + (c-b) \cdot \sum_{u \in [n]} \sum_{r \in [n]-u} (d_G(u,r) - 1)}{n^3 \cdot b + n}\right).$$

We can say that $d_G(u,r) < \Theta\left(\frac{2}{\sqrt{c+b}}\right)$, as player u would connect to player r otherwise. Player u would become closer to half the nodes on the path otherwise and reduce her betweenness cost through the routing potential gained by the link addition. Therefore we have,

$$\rho(G) = \mathcal{O}\left(\frac{|E(G)| + n^3 \cdot b + n^2 \frac{c-b}{\sqrt{b+c}}}{b \cdot n^3 + n}\right).$$

It follows that

$$\mathcal{O}\left(\frac{n^3 \cdot b}{n^3 \cdot b + n}\right) = \mathcal{O}(1),$$

and

$$\mathcal{O}\left(\frac{n^2 \frac{c-b}{\sqrt{b+c}}}{n^3 \cdot b + n}\right) = \mathcal{O}\left(\frac{c-b}{n^2 \cdot b + 1}\right) = \mathcal{O}\left(1\right),$$

as $c + b \geq \frac{1}{n^2}$ and $c < 1$. Thus, it only remains to consider $\mathcal{O}\left(\frac{|E(G)|}{b \cdot n^3 + n}\right)$.

As $|E(G)| = \mathcal{O}(n^2)$ for any Nash equilibrium, we have

$$\rho(G) = \mathcal{O}(n).$$

□

Chapter 5

Related Work

With their introduction in 2013 by Spilman [20], the study and implementation of payment channels have gained traction in the last years. While Spilman envisioned a unidirectional channel originally, the focus has moved towards bidirectional channels [11, 7, 13]. Generally, to set up a channel, a joint account between the two parties of the channel is created on-chain. Using this channel, the parties can exchange funds off-chain, updating the state of the channel with signed transactions. Together, these channels can build a network, used to route off-chain transactions via existing channels. Payment channels networks have already become a reality. The Lightning network, envisioned by Poon and Dryja [7], currently functions as Bitcoin's [1] Layer 2. Further, Raiden [21] operates as Ethereum's [4] payment channels network. Our work is independent of the channel construction specifications and thus applies to all such solutions.

Algorithmic payment channels network design by a central authority is studied by Avarikioti et al. [22]. Given a set of transactions, the optimal graph structure and fee assignment, maximizing the profit of the central authority, is analyzed. Furthermore, Avarikioti et al. [23] investigate the online and offline computation of a capital-efficient payment channels network. Our work, on the other hand, studies decentralized payment channels network design.

Fabrikant et al. [14] introduced network creation games in 2003. In their game, referred to as sum network creation game, a player unilaterally creates links to minimize the sum of distances to other players in the network. Albers et al. [24] build on these results, improving the upper bound for the price of anarchy and studying a weighted network creation game. While these papers solely focus on a player's closeness centrality, our model also includes players betweenness centrality.

In subsequent work, network creation games were expanded to various settings. The idea of bilateral link creation was introduced by Corbo and Parkes [25]. Demaine et al. [26] devise the max game, where players try to minimize their radius. Intrinsic properties of peer-to-peer networks are taken into account in the network creation variation conceived by Moscibroda et al. [27]. The idea of bounded budget network creation games was proposed by Ehsani et al. [28]. In bounded budget network creation games, players have a fixed budget to establish links. Nodes strive to minimize their stretch, the ratio between the distance of two nodes in a graph, and their direct distance. Àlvarez et al. [29] introduce the celebrity game, where players try to keep influential nodes within a fixed distance. However, the objectives in all these games give little insight to the control a player has over a network. This control is desired by players in Layer 2 to maximize the fees received for routing transactions.

A bounded budget betweenness centrality game was introduced by Bei et al. [30]. Given a budget to create links, players attempt to maximize their betweenness centrality. Due to their complexity, betweenness network creation games yield limited theoretical results, in comparison to those of the sum network creation game, for instance. In contrast to our work, a players closeness centrality is not taken into account. Thus, not providing insight into a player's distance to others in comparison to our model.

Buechel and Buskens compare betweenness and closeness centralities in [31]. However, their analysis is not in a network creation game setting, as their notion of stability does not lead to Nash equilibria. We, on the other hand, study the combination of betweenness and closeness incentives in a network creation game setting.

Chapter 6

Conclusion

We introduced a network creation game combining betweenness and closeness incentives for a player; modeling the objectives of a player in a payment channels network. Our game encapsulates both a player's willingness to act as a go-between for transactions and want to minimize her cost for making transactions over the payment channels network.

The social optimum for our model is identified for the entire parameter space. Additionally, we studied if and when prominent graphs are Nash equilibria; providing insight to the structures emerging when players act selfishly with betweenness and closeness centrality objectives. These results are backed up by simulations.

Finally, we upper bounded the price of anarchy of the game in the parameter space — furthering the understanding of the lack of degradation in Layer 2 if players act in an egocentric manner.

6.1 Future Work

While we upper bound the price of anarchy for the entire parameter space, the upper bound we find for the price of anarchy when $b + c \geq \frac{1}{n^2}$ and $c < 1$ is not necessarily tight. Thus, it would be interesting to see whether this bound can be improved.

In our model, the channel is set up unilaterally by a single node. The game could be adapted to a fractional model, allowing two parties of a channel to divide the cost between them, as is the case in most payment channels network. In close relation, our model fixes the capital placed in the channels. Allowing nodes to decide on the amount of capital deposited in the channels would be another possible extension.

Additionally, to more accurately describe the objectives of players in a payment channels network, weights could be added to the model. For one, weights associated with players in the closeness term would represent a player's inclination to connect with others. Players with whom she is likely to exchange transactions would be associated with higher weights in the closeness term than those players she will hardly ever exchange transactions. For betweenness centrality, weights would be related to pairs of nodes, expressing the likeliness of transactions being exchanged between them.

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Appendix A

Parameter Sweeps

We show some parameter sweeps for the weights b and c generated by our simulation. These show when some of the prominent graphs analyzed in Chapter 4.2 are Nash equilibria.

A.1 Complete Graph

Figure A.1 shows the simulation results for the complete graph. Here the underlying assumption was made that lower ID players connected to all higher ID players. However, independent of this assumption the simulation yields the same results.

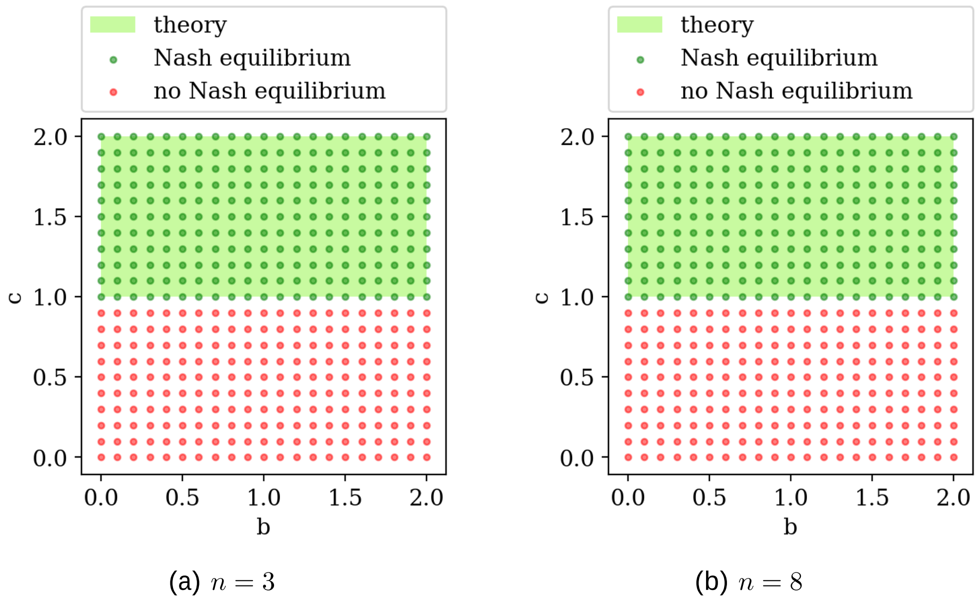
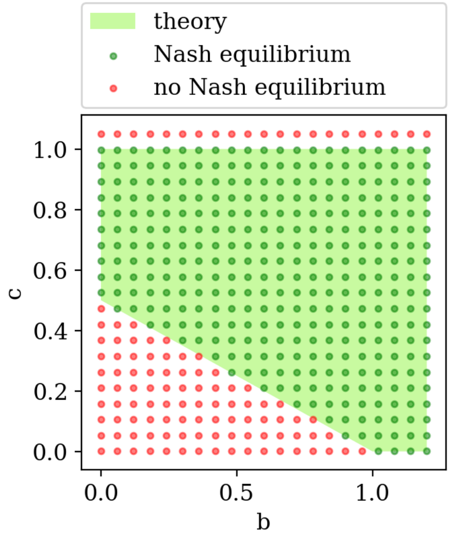


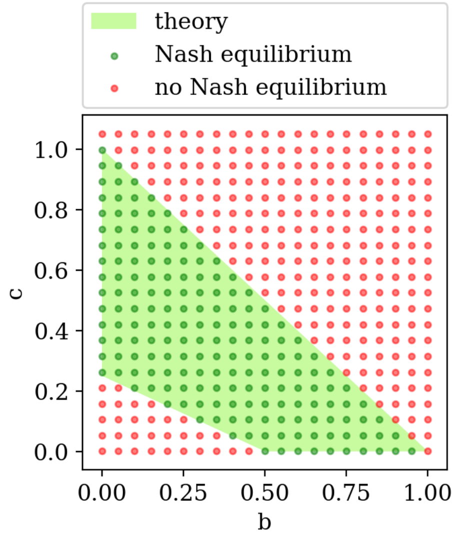
Figure A.1: Parameter map for complete graph.

A.2 Circle Graph

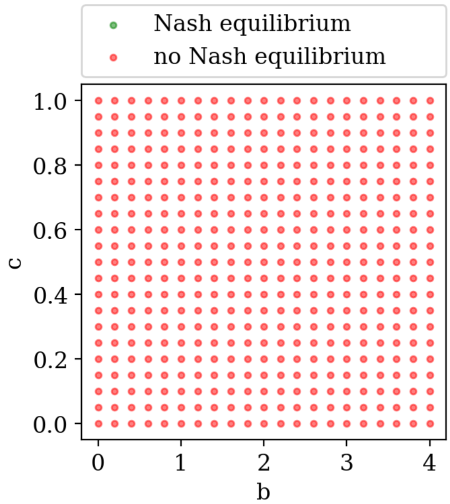
Simulation results for the circle graph are shown in Figure A.2. Here all players have exactly one outgoing link, as the bounds for this case are less restrictive than if any player would have two outgoing links.



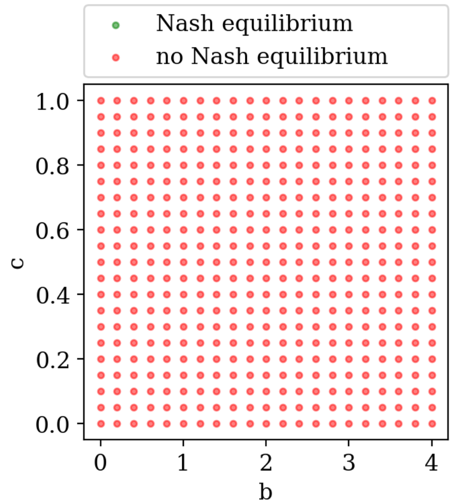
(a) $n = 4$



(b) $n = 5$



(c) $n = 6$



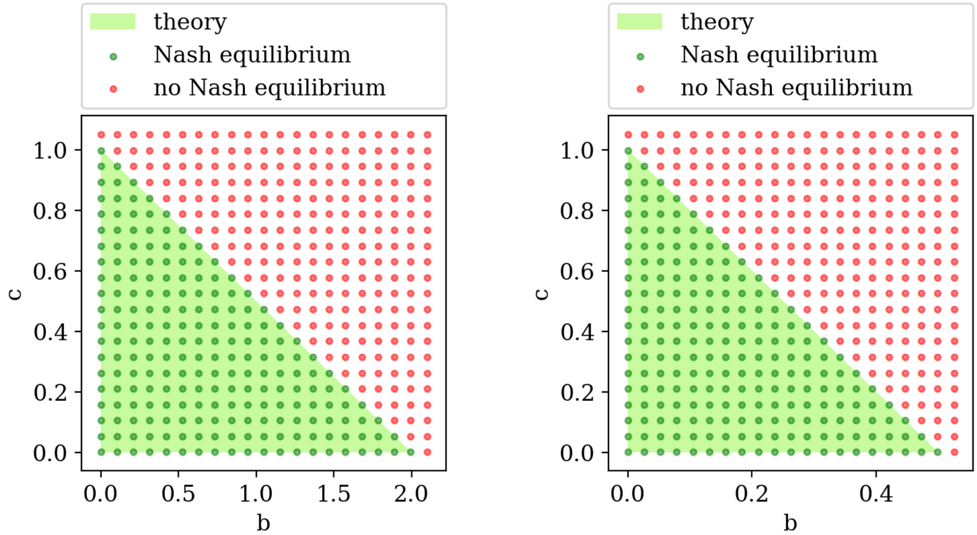
(d) $n = 7$

Figure A.2: Parameter map for circle graph.

For $n = 4$ and $n = 5$, the simulation matches the theory. Additionally, the simulation also suggested that for $n \geq 6$ the circle graph is never a Nash equilibrium as indicated by the results in Figures A.2c and A.2d.

A.3 Star Graph

In Figure A.3 we show when the star is a Nash equilibrium. For the simulation one player connects to everyone else. Other possible star graphs have more restrictive bounds on b and c.

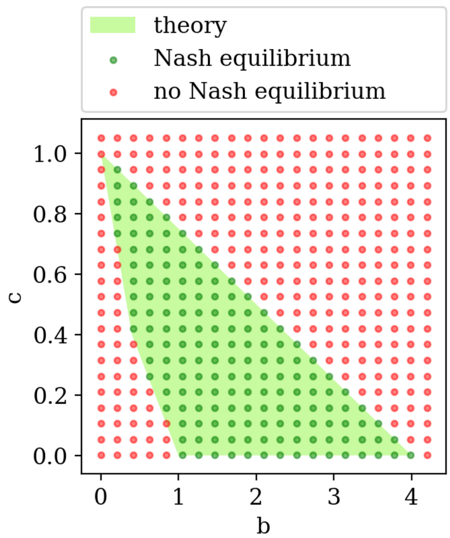


(a) $n = 4$ (b) $n = 7$

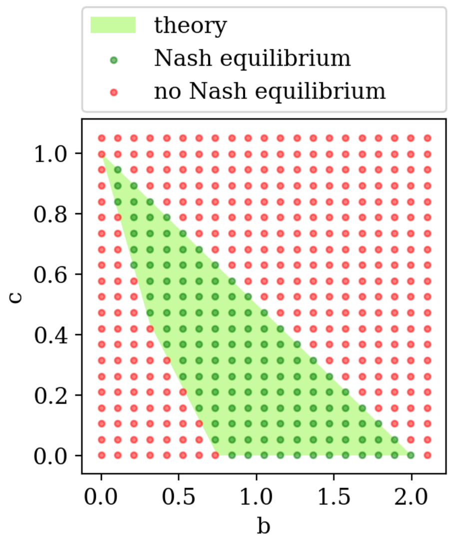
Figure A.3: Parameter map for star graph.

A.4 Complete Bipartite Graph

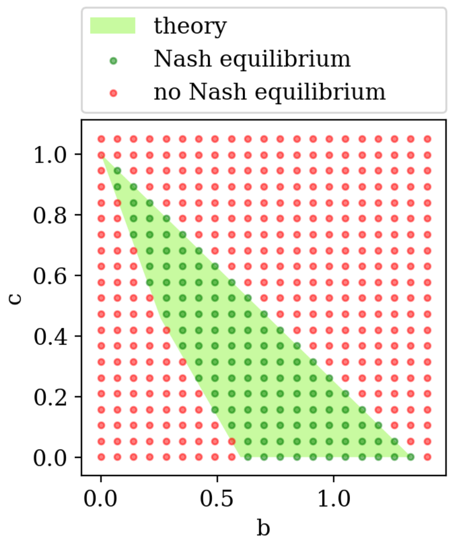
Figure A.4 shows when the complete bipartite graph is Nash equilibrium for the presented parameters. The simulation was done for players in the smaller subset having all the outgoing links, as this case leads to less restrictive bounds for b and c.



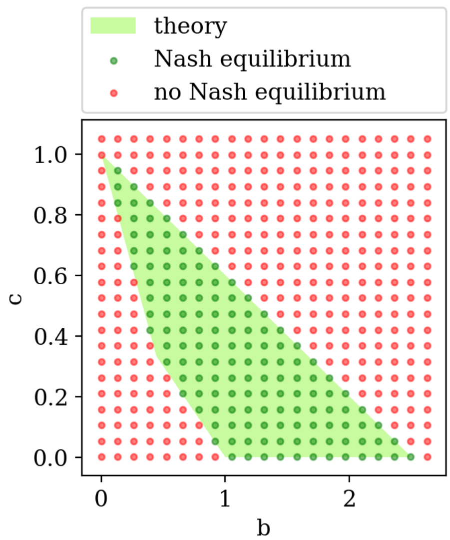
(a) $n = 6$, $r = 3$, $s = 3$



(b) $n = 7$, $r = 3$, $s = 4$



(c) $n = 8$, $r = 3$, $s = 5$



(d) $n = 8$, $r = 4$, $s = 4$

Figure A.4: Parameter map for complete bipartite graph.