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Distributed Computing

Over-bidding Strategies in Combinatorial Auctions

Master's Thesis

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Abstract

Combinatorial auctions (CAs) allow an auctioneer to match their goods to bundles, individually composed by the bidders, at competitive prices. CAs gained importance as tools for auctioning ranges (like time slots or frequency licenses). This Master's thesis studies over-bidding in CAs with single-minded bidders. First, we confirm that there are no over-bidding strategies for non-decreasing payments. Then we focus on the widely used quadratic payment rule – which is not non-decreasing – and aim to find a minimal example of over-bidding. We developed the notion of CA classes, mapping CAs with equivalent behavior onto a graph. Therefore, we are able to list CA classes by increasing number of bidders. Based on an analysis of the CA class represented by star-graphs, we prove that CA classes with a single effective core constraint are robust against over-bidding. This result helped us to show that for four or fewer bidders, there are no over-bidding strategies in single-minded CA. We list the candidate CA classes on five bidders, for which over-bidding strategies could exist, and conclude with an analysis of two candidate classes.

Contents

Chapter 1

Introduction

1.1 Motivation

Just this week, Paul Milgrom and Robert Wilson where honored with the Nobel Memorial Prize in Economics for their work on auction theory. Auctions are well established mechanisms to sell unique goods with unknown value at a competitive market price. Combinatorial Auctions (CAs) are auctions which allow the bidders to bid on bundles (sets of items), instead of being limited to place bids on single items. The CA Mechanism is given by a payment function and an allocation algorithm, which are evaluated on the placed bids. The auctioneer collects all bids and then determines the winners of the auction according to the allocation algorithm. The allocation states which bids are winning under the following two constraints.

- For every bidder, none or a single bid placed by this bidder are winning.
- Every item may belong to at most one winning bundle corresponding to a winning bid, or simply put - every item can be sold exactly once.

Then the winning bidders acquire their winning bundle in exchange for a payment dictated by the payment function. By bidding on sets of items, CAs let bidders express complex valuations. For example a completed collection may be worth more than the summed values of the individual pieces it contains. Or there might be diminishing returns when adding interchangeable items to a bundle.

CAs are especially well suited when auctioning off continuous ranges, like time slots for ads or radio spectrum licenses. CAs have found widespread use in practice, some applications amount to multi-billion dollar business [1, 2]. This sparks the interest to have well understood CA Mechanisms, which adhere to certain game theoretical properties. There is tension between providing the right incentives to the bidders, such that they don't attempt to manipulate the auction, while maximizing the reported social welfare (sum of winning bids) among the bidders and maximizing the revenue of the auctioneer.

For the widely used quadratic payment-rule [3], examples of profitable over-bidding have been noted [4]. Specific circumstances allow a bidder to favorably manipulate the auction by placing an over-bid greater than the individual private value of the bundle. *Under*-bidding strategies (bid shading) are not problematic in the same way. In a competitive environment a bidder risks not acquiring a bundle when bidding too low. Over-bidding on the other hand is not self-regulating. Over-bidding can be damaging to the other bidders and the auctioneer as well as reduce the trust in the auction mechanism as a whole.

1.2 Objective

In this Master's thesis we study over-bidding strategies in CAs. We aim to understand what properties of valuation lead to profitable over-bidding. This knowledge could then be used to design CA mechanisms which cannot be manipulated by over-bidding.

We begin by studying the existence of over-bidding strategies. In general CAs there are examples of over-bidding strategies with just three bidders. When placing multiple bids, over-bidding a losing bid can potentially lower the payment of ones winning bid. On the other hand, in single-minded CAs, where each bidder places exactly one bid, the sole example of profitable over-bidding known to us encompasses eleven bidders. This is a quite complex example, due to the NP-hard nature of the optimization problems of winner determination and core-selecting[1] payment. This thesis focuses on single-minded CAs, with the aim to find a minimal example (w.r.t. the number of bidders) of over-bidding. We strive to prove that no over-bidding strategies exist for CAs with up to n players before considering examples with $n + 1$ players.

The search for a minimal example of over-bidding in single-minded CAs should help us understand what factors are essential to the existence of profitable over-bidding. Further, we hope to learn about CA scenarios where no over-bidding strategies exist.

[1] The core property will be formally introduced at definition 2.7. It requires that after the auction based on the reported bids there is no subset of bidders which can mutually improve their utility by trading acquired items among themselves.

Chapter 2

Preliminaries

We study auctions from the perspective of Game theory. The base assumptions are that each player (the bidders and the auctioneer) are independent, rational and selfish. Each player aims to maximize their personal utility, by rational decision making based on their respective incomplete knowledge. Let us begin by introducing the first price sealed-bid auction, also known as blind auction. It will serve as a comparison to the combinatorial auction (CA).

2.1 Blind auction

Given a set of M goods (items) to be auctioned off and and a set of N bidders (players/participants). For every item $j \in M$, each bidder $i \in N$ has a private valuation $v_i(j) \in \mathbb{R}_{\geq 0}$. The valuation represents the true value bidder i would have from owning item j. This information is considered private and is not available to the other players. In every round one item is sold as follows. Each bidder places a sealed bid on the item. The auctioneer collects all bids and selects the highest bidder. The highest bidder pays the submitted price (first price) in exchange for the item. That way the bids that didn't win remain hidden information to the other bidders.

2.2 Combinatorial Auction (CA)

Given a set of M goods and N bidders. For each bundle of goods $K \subseteq M$, each bidder i has a secret value $v_i(K) \in \mathbb{R}_{\geq 0}$ and may submit a bid $b_i(K) \in \mathbb{R}_{\geq 0}$. We can choose to limit every player to bid on at most r bundles. If $r = 1$ we call the auction single-minded, for $r > 1$ multi-minded. This Master's thesis studies single-minded CAs (SMCA). For simplicity the definitions have been adapted accordingly. There's a single round of bidding where every participant $i \in N$ places one bid b_i. The bid b_i declares which bundle player i is willing to purchase, and the bid value represents the highest price player i is willing to pay for this bundle. The bid profile $b = (b_1, ..., b_n)$ denotes all bids placed in the auction.

A CA mechanism $\mathcal{M} = (X, P)$ is given by a winner determination algorithm X and a payment function P. The winner determination selects the winning bundles. Every winning bundle is awarded to the player placing the corresponding winning bid in exchange for the payment determined by the payment function. As Game theory dictates, each bidders goal is to maximize their own utility. The utility u_i for player i is given by the private valuation of the assigned bundle minus the corresponding payment: $u_i(b) = v_i(X(b)) - p_i(b, X(b))$. Both the valuation v_i and the payment p_i depend on the allocation algorithm X and the bid profile b. We first introduce the allocation algorithm X then the payments P. The social optimum is given by the allocation that maximizes the sum of valuations of the winning bundles. However the valuations are private, so the best the auctioneer can do is to maximize the reported social welfare.

Definition 2.1. Let $W(b, x) = \sum_{i=1}^{n} b_i(x_i)$ be the **reported social welfare** achieved by an allocation x. Let $W(b_{-i}, x_{-i}) = \sum_{j \neq i} b_j(x_j)$ and $W(b_L, x_L) = \sum_{j \in L} b_j(x_j)$. Let $X_L(b_L)$ be the set of allocations x maximizing the social welfare of bids in $L \subseteq N$, $W(b_L, x_L)$.

Definition 2.2. Allocation (winner determination) $X(b)$ is an allocation algorithm which returns efficient allocations x.At the i-th position the allocation x indicates if the i-th bid (and hence the i-th player) is winning $x_i = 1$ or losing $x_i = 0$. An allocation is said to be efficient if it maximizes the reported social welfare: $\max_x W(b, x) = \max_x \sum_{i=1}^{n} b_i(x_i)$. This optimization problem is subject to the following two constraints.

- Every bidder can be assigned one or none of its bids as a winning bundle.
- Every item can be sold once in its entirety, or else it will remain unallocated.

Every efficient allocation $x \in X$ is realized with the same probability, i.e. in cases of ties x is selected uniformly at random among all efficient allocations.

This thesis examines over-bidding in single-minded combinatorial auctions (SMCA). To do so we introduce a definition of over-bidding strategies and notation.

Definition 2.3. Over-bidding strategy: Considering bidder i with valuation $v_i(K)$ for bundle K, and fixed bids of all other player b_{-i}. An over-bidding strategy exists for player i on bundle K if there is some bid value $b_i^o > v_i$, such that the resulting utility of the over-bid placed on bundle K is strictly greater than the utility of the truthful bid b_i^v (on the same bundle). To shorten notation we write $b^o = (b_i^o, b_{-i})$ and $b^v = (v_i, b_{-i})$. Formally we say that there exists an over-bidding strategy if there is at least one participant i who, for some bids b_{-i}, can increase the utility u_i by over-bidding on some bundle rather than bidding their true valuation:

$$u_i(b^o) = v_i(X(b^o)) - p_i(b^o, X(b^o)) > v_i(X(b^v)) - p_i(b^v, X(b^v)) = u_i(b^v)$$

With simplified notation we express that there is an over-bidding strategy if for some $i \in N$ it holds that $u_i(b^o) > u_i(b^v)$.

We say a CA mechanism $\mathcal{M} = (X, P)$ is robust against over-bidding strategies if there exist no over-bidding strategies in $\mathcal{M}$. We will study the existence of over-bidding strategies in single-minded CA, and consider the following payment rules.

2.3 Payment-functions and related properties

Definition 2.4. Voluntary participation holds when no bidder risks an outcome resulting in a negative utility when bidding truthfully. In other words, losing bidders pay zero and winning bidders are never required to pay more than the placed bid; $\forall i \in N : p_i \leq b_i$.

We only consider payments which satisfy voluntary participation.

Definition 2.5. First price payment: The winning bidders payment is equal to the corresponding winning bid; $\forall i \in N : p_i(b, x) = b_i(x_i)$.

Right from the start we can state that there are no over-bidding strategies for the first price payment. If an over-bid is not winning, the utility and payment are zero. A winning over-bid on the other hand leads to negative utility of $u_i = v_i - b_i^o$, as by definition $b_i^o > v_i$. Sounds like the first price payment has all the properties we are looking for! Yes and no, the first price payment has another issue. A truthful bid always results in zero utility, for a losing as well as a winning bid. Hence the only way to generate a positive utility for a bidder is to under-bid. In practice the first price payment is not used for CA. The lacking incentives to bid truthfully lead to a worse seller revenue when compared with other core-selecting payments. We will use the first price payment as a reference point – an upper-bound of the other payments.

The Vickery-Clarke-Groves (VCG) payment, is the unique payment rule which results in a truthful CA mechanism with allocation algorithm X. We denote the VCG-payment as p^V, and for a specific bidder i, we write p_i^V.

Definition 2.6. The **VCG-payment** is given by the maximal reported social welfare excluding i's bid minus the reported social welfare of the winning allocation excluding the i-th bid: $x_{-i} = x \setminus x_i$

$$p_i(b, x) := W(b_{-i}, X_{-i}(b_{-i})) - W(b_{-i}, x_{-i})$$

The VCG-payment p_i^V is a measurement of player i's contribution to the solution. It represents the difference of the best solution would i not participate to the remainder of the solution after removing i's share. Note that the own bid b_i has no influence on the payment p_i^V.

Definition 2.7. The **core** is the set of all points $p(b,x)$ which satisfies the following constraint for every subset $L \subseteq N$:

$$\sum_{i \in N \setminus L} p_i(b,x) \geq W(b_L, X_L(b_L)) - W(b_L, x_L)$$

A payment is called core-selecting if the payment point always lies within the core.

The core is described by lower bound constraints on the payments such that no coalition can form a mutually beneficial renegotiation among themselves. Those core constraints impose that any set of winning bidders must pay at least as much as their opponents would be willing to pay to get their items. The first price payment always lies within the core. In fact it's an upper-bound of the core.

Definition 2.8. The **minimum revenue core** forms the set of all points $p(b,x)$ minimizing $\sum_{i \in N} p_i(b,x)$ subject to being in the core.

2.3.1 Core-selecting payment functions

Let us look at an example to better understand the core and the various payment functions we will define.

Example 2.9. We consider a single-minded combinatorial auction (SMCA) with three bidders and two items. We name the bidders A, B, C and the items 1,2. Player A bids 7 on the single item 1, player B bids 5 on item 2 and player C bids 9 on the bundle containing both items 1 and 2 (table 2.1). We note the bid-profile $b = (7,5,9)$. For those bids we consider the payment outcome $p(b)$ for various payment functions p. First we evaluate the core. To do so we determine the winners, which are A and B. Selling the items individually to A and B amounts to a reported social welfare of 12, whereas the second best solution sums to 9, when selling both items to bidder C. The core constraint on the winning bidders payment is $p_A + p_B \geq 9$, as player C is willing to pay 9 for the items. All other core constraints are less restrictive, they don't further describe the core. Thus, combined with the upper-bound by the first price payment the core is given as the green triangle drawn in figure 2.1.

Definition 2.10. The **VCG-Nearest** payment rule (quadratic payment, VCGN) picks the closest point to the VCG-payment within the minimum-revenue core with respect to euclidean distance.

For the bid profile b taken from example 2.9, the VCG payment for player A, equals 4: the second best solution when A does not participate has a reported social welfare of 9. Then we subtract 5, the sum of other bids in the winning

bidder	bundle	bid b	x	First price	p^V	VCGN	Proxy	Prop
A	$\{1\}$	7	1	7	4	5.5	4.5	$21/4$
B	$\{2\}$	5	1	5	2	3.5	4.5	$15/4$
C	$\{1,2\}$	9	0	0	0	0	0	0

Table 2.1: Example 2.9, the outcomes for the various payments.

allocation. VCG payment for player B is 2. The sum of VCG payments is equal to six. The core requires that $p_A + p_B \geq 9$, thus the VCG-Nearest rule closes the distance to the core by adding each of the winning bidders $\frac{3}{2}$ to their payments.

Definition 2.11. The **Proxy payment** selects the point in the core with minimal euclidean distance to the origin (zero payments) for winning bids. Losing bids have a payment of zero. Equivalently we can define it as the unique point in the core being of the form $p_i(b, x) = min[\alpha, b_i]$ for some $\alpha \geq 0$.

In example 2.9 the proxy payment is 4.5 for each of the two winning players. This corresponds to the closest point in the core from the origin.

Definition 2.12. The **Proportional payment** for winning bids, is given by the point within the core that minimizes the total payment $\sum_{i \in N} p_i(b, x)$ and is of the form $p_i(b, x) := \alpha \cdot b_i(x_i)$ for some $\alpha \geq 0 \leq 1$. Losing bids have a payment of zero.

Continuing example 2.9, the proportional payment $(p_A, p_B) = (\frac{21}{4}, \frac{15}{4})$ lies at the intersection of the diagonal, connecting the origin to the first price point, with the core constraint.

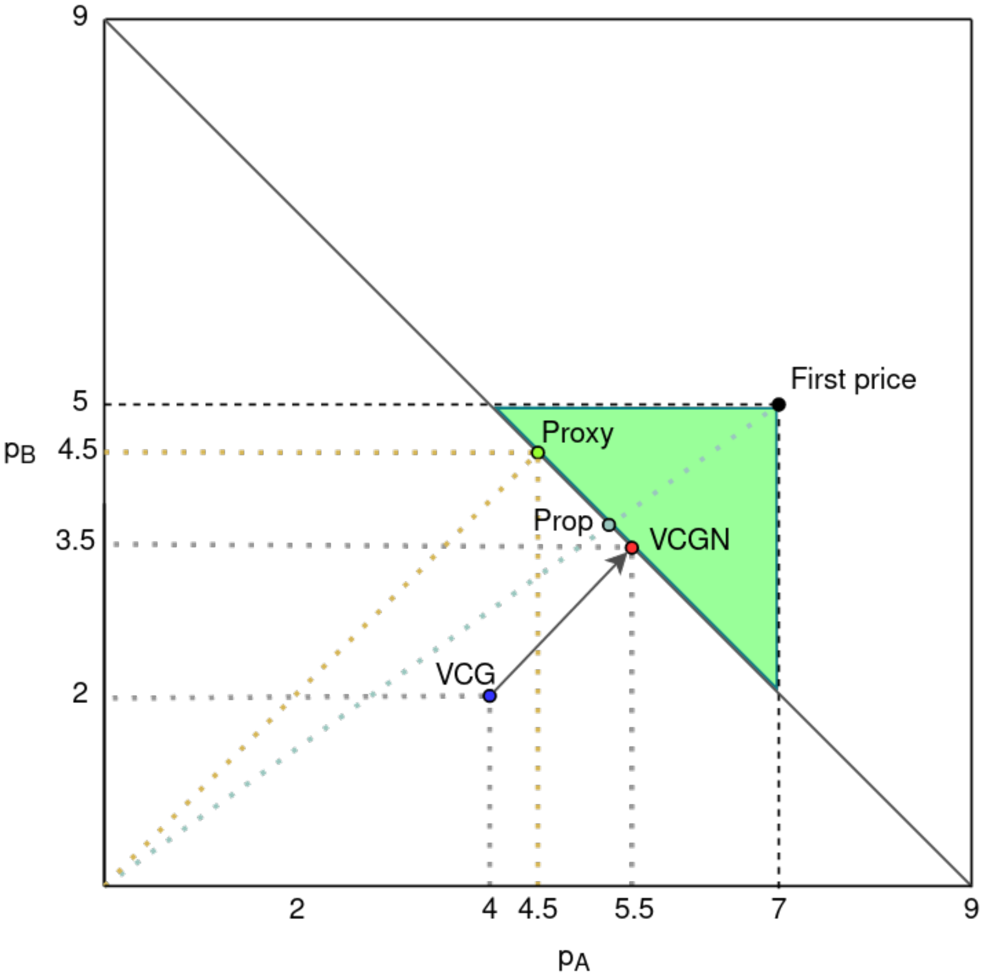

Figure 2.1: SMCA example 2.9 illustrates the core (green triangle), and various payment points: first price, proxy, proportional (Prop), VCG-Nearest and VCG payment.

2.4 Computation

Evaluating the outcome of CAs for specific bids is rather complex to do by hand. In general winner determination as well as determining the core-selecting payments are NP-hard problems. To solve examples and validate assumptions we coded a solver in C++ using the CGAL library [5]. The code can be found in gitlab with this URL: `https://github.com/meiera-CAOS/CAOS`. Here's a short overview of the implemented functions:

- winner determination (LP / bruteforce)
- core allocation (LP / bruteforce)
- payment functions
  - fist price
  - VCG
  - VCG-Nearest (QP)
  - Proxy (QP)
  - Proportional (limited precision binary search over LP)
- print functions to output bids, winner determination, payments

In brackets noted are the techniques used for the computation. LP, QP stand for linear program and quadratic program respectively. A cleaner implementation of winner determination would be to use integer programming, which only allows for integer solutions. The implementation was tested by comparing the outputs to known examples from research[1].

2.5 Examples of over-bidding

In this section we give two examples of over-bidding in CAs, one for multi-minded bidders and one for single-minded bidders.

2.5.1 Over-bidding in multi-minded CA

The example on multi-minded bidders for the VCGN-payment is based on an example from the paper [4]. The auction consists of three bidders A, B, C and two items 1,2. The truthful bid-profile b^v is given as follows: A bids 4 on the bundle $\{1\}$, B bids 4 on $\{2\}$ and 5 on $\{1, 2\}$, C bids 6 on $\{1, 2\}$. The over-bidding bid-profile b^o is identical to b^v, except that B over-bids 7 instead of 5 on the global bundle $\{1, 2\}$. In both cases the maximal reported social welfare is equal to 8 where bidder A and B acquire a single item each. Therefore the bids of 4 on an individual item are the winning bids. The payment outcomes are illustrated in figure 2.2. By over-bidding the losing bid on the global bundle, bidder B increased the VCG-payment of player A from 2 to 3. This results in a decrease of the over-bidders VCGN payment p_B by half a unit. Note that the winning bid of player B remaines unchanged.

[1]The tests are listed in the testsets folder of the aforementioned git repository.

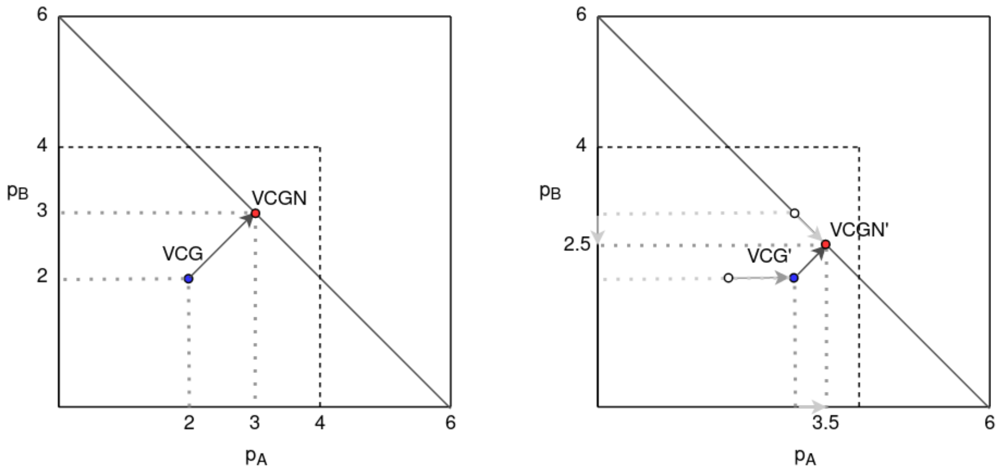

Figure 2.2: On the left, the payment outcome for the truthful bid-profile b^v, on the right the changes due to the over-bid in b^o.

2.5.2 Over-bidding in single-minded CA

In single-minded CA, as every bidder places exactly one bid, a bidder cannot manipulate the auction by misrepresenting one of their losing bids. Hence every over-bidding strategy is limited to over-bidding the winning bid. The over-bidding example in SMCA with VCG-Nearest payment shown in table 2.2 motivates our search for a minimal example. We are not aware of an example on less than eleven bidders.

This example is quite complex, we can see this by looking at the core. Every one of the 2^{11} possible subsets of bidders adds a core constraint which might impact the payment outcomes. Further, every core constraint depends on the optimal winner determination within the subset - an NP-hard problem on it's own. To analyze over-bidding for the VCGN payment in SMCA, examples with less bidders would reduce the complexity considerably. If we had an example of over-bidding which is proven to be minimal, this could help crystallize the essential conditions where over-bidding strategies exist. Also it would imply that all auctions with fewer bidders are robust against over-bidding.

bidder	bundle	bid b^v	over-bid b^o	$p^V(b^v)$	$p^V(b^o)$	$p_{VCGN}(b^v)$	$p_{VCGN}(b^o)$
#1	$\{1\}$	5	5	2	1	37/12	36/12
#2	$\{2\}$	5	5	0	0	16/12	18/12
#3	$\{\mathbf{3}\}$	**4**	**5**	**1**	**1**	**37/12**	**36/12**
#4	$\{4\}$	1	1	0	0	7/12	6/12
#5	$\{5\}$	1	1	0	0	7/12	6/12
#6	$\{6\}$	1	1	0	0	10/12	12/12
#7	$\{1,2,4\}$	5	5				
#8	$\{2,3,5\}$	5	5				
#9	$\{1,3,6\}$	7	7				
#10	$\{4,5,6\}$	2	2				
#11	$\{2,3,4\}$	5	5				

Table 2.2: Over-bidding example in SMCA on eleven bidders from Bosshard18 [4]. Bidder #3 over-bids their bid on bundle $\{3\}$, causing a decrease in the VCG-payment (p^V) of bidder #1 and a decrease in the over-bidders VCG-Nearest payment (p_{VCGN}).

Chapter 3

Related work

For spectrum auctions[1] the benefits of combinatorial auctions over classical auction mechanisms have been studied. Most notably, CA allow "alternative technologies that require the spectrum to be organized in different ways to compete in a technology-neutral auction." [1]

The computational complexity of CAs has been addressed as an initial hurdle to implement CA mechanisms in practice [6]. Over the years, increased computational power and more efficient algorithmic solutions, coupled with some complexity limiting design choices, allowed the effective use of CAs. The impact w.r.t. efficiency and revenue of the two main design choices – compact or fully expressive bids and first price or core-selecting payments – has been analyzed [7]. The authors argued in favor of simplicity, both in the expressiveness of bidding and the complexity of payment.

Winner determination considers feasible allocations of bundles which maximize the reported social welfare. One branch of combinatorial auctions research focuses on how various payment functions affect the behaviour of the bidders when combined with reported social welfare maximizing allocation algorithms. The Vickery-Clarke-Groves (VCG) payment is the unique payment function which then results in a truthful mechanism. Truthful mechanisms incentivize the bidders to bid according to their private valuation. The incentive is given as for each bidder, a truthful bidding strategy results in maximal utility. The VCG-payment is not suitable in practice due to various shortcomings [8]. The most pronounced issue of the VCG-payment may be that it frequently leads to very low (even zero) seller revenues.

To give guarantees on the seller revenue payment rules which select points within the core [9] are used. Drawing the parallels to stable matching mechanisms, Day and Milgrom [10] analyzed the benefit of core-selecting CA in 2007. They note that minimal-revenue core-selecting payment maximizes the truthful bidding. There are various core-selecting payment functions, each optimizing a

[1]Auctions used by governments to sell signal transmission licences on specified frequency ranges of the electromagnetic spectrum.

point within the core according to its own criteria.

Variations of the quadratic core-selecting payment [3] are generally used in practice for spectrum auctions with core-selecting payment [11]. The quadratic rule picks the point within the core with least euclidean distance to the VCG-payment[2]. The authors hypothesize that this should preserve as much truthful bidding incentive as possible. Other core-selecting payments have been suggested and trade-offs between the payments are being analyzed. In 2018, driven by a computational search for good core-selecting rules, two alternative payment functions have been proposed [12]. Those rules were classified as Large-style rules, as they provide better incentives for bidders with large values compared to the quadratic rule. Since core-selecting auctions are not truthful, the bidders action space cannot be simplified to truthful bids, making the outcomes increasingly hard to predict. This has motivated the study of Pure-Nash-Equilibria and Bayes-Nash-Equilibria for simple auction settings like the LLG model [11, 13]. Incentives for over-bidding [14] have been described for multi-minded bidders. Over-bidding losing bids was found to be profitable, when increasing other bidders payments links to a decrease of the over-bidders payment. Further, equilibria in minimal-revenue core-selecting auctions including over-bidding were discovered. The analysis was conducted for multi-minded bidders for full information and a variety of incomplete information settings. Positive impacts of over-bidding were noted to be increased seller revenue and improved expected efficiency. An example[3] of over-bidding in single-minded (one bid per bidder) CA was described [4] for quadratic payment. To prevent over-bidding by design, the notion of non-decreasing payment rules [4] was introduced. In single-minded CAs non-decreasing payments do not suffer from over-bidding, as non-decreasing payments prevent higher bids leading to lower payments. The quadratic rule is not non-decreasing. For the quadratic rule, it remains unclear under what conditions over-bidding strategies are possible.

[2] That's why we refer to the rule as the VCG-Nearest payment.

[3] The example is transcribed in table 2.2

Chapter 4

Non-decreasing payments

The over-bidding examples in section 2.5 have shown us that the utility gain of over-bidding strategies is linked to a decrease in the over-bidder's payment. A natural way to counteract this is with payment-functions for which an increase in a single bid never reduces the corresponding payment. The paper by Bosshard et al. [4] defines non-decreasing payment rules for CA as follows.

Definition 4.1. For any allocation x let $\mathcal{B}_x$ be the set of bid profiles for which x is efficient. The payment-rule $p(b, x)$ is **non-decreasing at** x if, for all bidders i and bid profiles $b, b' \in \mathcal{B}_x$ with $b_{-i} = b'_{-i}$ the following holds:

$$b'_i \geq b_i \implies p_i(b', x) \geq p_i(b, x)$$

A payment rule $p(b, x)$ is **non-decreasing** if it is non-decreasing at all allocations x.

As long as the over-bid does not cause a change in the allocation x, any non-decreasing payments do not allow for profitable over-bidding strategies. This follows directly from the fact that in those circumstances an over-bid can not result in a reduced payment.

Let us begin our analysis of over-bidding strategies in single minded combinatorial auctions (SMCA). We split the over-bidding scenarios into two cases.

- If a truthful bid would be losing, can a winning over-bid return a positive utility?
- If a truthful bid is winning, can an over-bid increase the utility?

Note that the first case considers a change in allocation, the over-bidding player i turns a losing bid into a winning bid. For the second case however, we consider a fixed allocation x. The allocation x remains efficient when increasing the bid of one of the winning players. Therefore we will be able to reason with non-decreasing payments in the second case.

4.1 Over-bidding losing bids

Here we show that it is not possible to get a positive utility by over-bidding a bid, where the truthful bid would otherwise not be part of the allocation.

Theorem 4.2. *Given a CA mechanism where X is an allocation algorithm maximizing reported social welfare, and P is a core-selecting payment satisfying voluntary participation. Consider N single minded bidders, for any bidder i and fixed bids of the other players b_{-i}. If a truthful bid of player i is a losing bid for $x \in X(v_i, b_{-i}), x_i = 0$, there are no over-bidding strategies for this player.*

Proof. A losing bid has zero utility due to voluntary participation, no bundle is acquired, no value gained and the payment is zero. A losing over-bid equally results in zero utility. Thus if there is over-bidding that increases the utility, it must result from winning over-bids. For player i where the truthful private value v_i is a losing bid, we observe that for the winner determination over all bids, the allocated solution's social welfare must be greater or equal[1] to the maximal social welfare achievable by any solution where i is winning. The over-bid of player i must be winning, thus leading to a solution with greater or equal social welfare than any allocation where i is losing. As notation we write x^i to be a maximal allocation where i is winning – independent of the bid value. For the truthful and over-bidding bid profiles we write $b^v = (v_i, b_{-i}), b^o = (b^o_i, b_{-i})$ in short. Then we get:

$$W(b^v, x^i) \leq W(b^v, X(b^v)) \leq W(b^o, x^i) = W(b^o, X(b^o))$$

The difference in social welfare between a maximal allocation and a maximal allocation containing i is $W(b^v, X(b^v)) - W(b^v, x^i)) = \Delta \geq 0$. We set the over-bidding value to be the minimal winning over-bid $b^o_i = v_i + \Delta$ which leads to $W(b^v, X(b^v)) = W(b^o, x^i)$. Note that it is sufficient to check if b^o_i can result in greater zero utility, as any over-bid greater b^o_i will be covered in the over-bidding a winning bid case (by assuming that $b^o_i = v_i$ is truthful).

Let T be the set of winning bids in $x' \in X(b^v)$ $T = \{j \in N \mid x'_j = 1\}$. Further let S be the set of winning bids in both x' and x^i, $S = \{j \in N \mid x'_j = 1 \wedge x^i_j = 1\}$. We look at the core constraint on the payments of $N \setminus T$ with the bids b^o under the allocation x^i to derive a lower bound.

[1] Equality of the social welfare would indicate a tie among multiple efficient allocations. The allocation algorithm X breaks ties by picking an efficient allocation x uniformly at random.

$$
\begin{aligned}
&\sum_{j \in N \setminus T} p_j(b^o, x^i) \\
\geq\ & \sum_{j \in T} b_j(X_T(b_T))_j - b_j(x^i_j) && \text{core property} \\
\geq\ & \sum_{j \in T} b_j(X_T(b_T))_j - \sum_{j \in S} b_j && \text{by definition of } S \\
\geq\ & \sum_{j \in T} b_j(x'_j) - \sum_{j \in S} b_j && \text{as } x' \text{ is efficient in } T \subseteq N \\
\geq\ & W(b^v, X(b^v)) - \sum_{j \in S} b_j
\end{aligned}
$$

In the last step we applied that the sum of the bids in T under the allocation x' is equal to the reported social welfare for b^v. This holds as x' is an efficient allocation for b^v and T contains all winning bids of x'.

Using the fact that a payment is never greater than the corresponding bid, we upper bound the same term that we lower bounded above.

$$
\begin{aligned}
&\sum_{j \in N \setminus T} p_j(b^o, x^i) \\
\leq & \sum_{j \in N \setminus T} b_j(x^i_j) \\
\leq & \sum_{j \in N} b_j(x^i_j) - \sum_{j \in S} b_j \quad \text{as } \forall j \in S : (x^i_j = 1) \\
\leq & W(b^o, x^i) - \sum_{j \in S} b_j
\end{aligned}
$$

Combined we get

$$
\begin{aligned}
W(b^v, X(b^v)) - \sum_{j\in S} b_j \leq \sum_{j\in N\setminus T} p_j(b^o, x^i) \leq W(b^o, x^i) - \sum_{j\in S} b_j \\
W(b^v, X(b^v)) = W(b^o, x^i) \quad \text{by choice of } b_i^o = v_i + \Delta \\
\implies W(b^v, X(b^v)) - \sum_{j\in S} b_j = \sum_{j\in N\setminus T} p_j(b^o, x^i) = W(b^o, x^i) - \sum_{j\in S} b_j \\
\implies \text{all inequalities above must infact be equalities, then} \\
\sum_{j\in N\setminus T} p_j(b^o, x^i) = \sum_{j\in N\setminus T} b_j(x_j^i) \\
\text{the only way to build that sum s.t. } \forall j \in (N \setminus T) \quad p_j \leq b_j \\
\text{is when } \forall j \in (N \setminus T) \quad p_j = b_j \implies p_i = b_i^o
\end{aligned}
$$

Therefore any over-bid b_i^o which changes a truthful losing bid to a winning bid results in utility $u_i^o = -\Delta \leq 0$. □

The proxy, proportional and VCG-nearest payments are each core-selecting and satisfy voluntary participation. If we pair any one of those payment functions P with a winner determination algorithm X maximizing the reported social welfare to get $\mathcal{M} = (X, P)$, then by theorem 4.2 the mechanism $\mathcal{M}$ is robust against over-bidding strategies of players where a truthful bid is a losing bid.

4.2 Over-bidding winning bids

We now consider winning truthful bids, that is to say for a player i and fixed bids of other players b_{-i}, the allocation $x \in X(v_i, b_{-i}), x_i = 1$ selects the i-th bid of value v_i as winning. Here we ask if for some over-bid $b_i^o > v_i$ the payment to acquire the corresponding bundle can decrease. When truthful bid v_i is part of the efficient allocation x, then for the over-bid b_i^o the allocation x must also be efficient. Further we note that for b^o, every efficient allocation must contain b_i^o, as the reported social welfare of every efficient allocation containing i was increased by the same amount the bid was increased $(b_i^o - v_i)$. If we assume an identical allocation $x \in X(b_i^o, b_{-i})$ for any over-bid b_i^o, then we can apply the non-decreasing payment functions definition 4.1 from the start of this chapter. That way, for non-decreasing payments it follows that the payment can not decrease for the fixed allocation x. This implies that over-bidding strategies do not exist for non-decreasing payments when over-bidding winning bids results in the same allocation x.

Theorem 4.3. *Given a CA mechanism $\mathcal{M} = (X, P)$ with an allocation algorithm X maximizing social welfare, and non-decreasing core-selecting payment function P. Considering single minded bidders, for any bidder i and fixed bids of the other players b_{-i}. If a truthful bid of player i is a winning bid for $x \in X(v_i, b_{-i}), x_i = 1$, there are no over-bidding strategies for this player under the allocation x.*

Proof. Considering some player i, we compare a truthful bid profile $b^v = (v_i, b_{-i})$, and an over-bidding profile $b^o = (b_i^o, b_{-i})$, $v_i \leq b_i^o$. Both the truthful bid and the over-bid are winning in the efficient allocation $x \in X(v_i, b_{-i}) \cap X(b_i^o, b_{-i}), x_i = 1$. A non-decreasing payment guarantees that the payment cannot decrease, $p(b^o, X(b^o)) \geq p(b^v, X(b^v))$. Hence, there are no over-bidding strategies. □

4.3 Over-bidding any bids

In this section we unite the results from over-bidding losing bids and over-bidding winning bids. We address the case of tie-breaking winner determination w.r.t. over-bidding and conclude with an observation about Nash equilibria.

What about the case when the allocation x' of the over-bid is different to the allocation x of the truthful bid $x \neq x', \quad x, \ x' \in X(b^v) \cap X(b^o)$? This can only happen in case of ties, i.e. when there are multiple efficient allocations. The winning probability of every efficient allocation is identical, due to the allocation algorithm X picking one efficient allocation uniformly at random among all efficient allocations. We can analyze the expected payment for each player and bidding profile. For non-decreasing payments it follows that the expected payment of any given bidding profile is non-decreasing, as for every distinct allocation the payment is non-decreasing. For non-decreasing payments the expected payment is non-decreasing. These two deductions imply that over-bidding is not profitable in expectation. We conclude that there are no over-bidding strategies in SMCA for non-decreasing, core-selecting payments which satisfy voluntary participation.

The corollary 4.4 follows from combining theorems 4.2 and 4.3 with the above insight on tie-breaking. When we compound the required properties on the allocation algorithm X and payment function P, those two theorems cover any over-bid in single minded CAs.

Corollary 4.4. *Given a CA mechanism $\mathcal{M} = (X, P)$, where X is an allocation algorithm maximizing reported social welfare, and P is a core-selecting, non-decreasing payment satisfying voluntary participation. For single minded bidders, there are no over-bidding strategies.*

We show corollary 4.4 by arguing that the expected utility of over-bidding is lesser equal to the expected utility of a truthful bid.

Proof. By theorem 4.2 we know that over-bidding losing bids leads to a utility of less or equal to zero. It is never profitable to over-bid a losing bid. Next we consider winning bids. Considering some bidder i, we compare a truthful bid profile $b^v = (v_i, b_{-i})$, and an over-bidding profile $b^o = (b_i^o, b_{-i})$, $v_i \leq b_i^o$, where the truthful bid is winning for some efficient allocations $x \in X(b^v), x_i = 1$. Let X^i be the set of efficient allocations for the bid profile b^v where player i is winning. Then the set X^i is equal to all efficient allocations for any over-bid b_i^o. The expected payment of a truthful winning bid is equal to

$$\mathbb{E}[p_i(X^i, b^v)] = \frac{\sum_{x \in X^i} p_i(x, b^v)}{|X^i|} \leq \frac{\sum_{x \in X^i} p_i(x, b^o)}{|X^i|} = \mathbb{E}[p_i(X^i, b^o)]$$

The inequality follows from theorem 4.3, which states that for any $x \in X^i$ the payment is non-decreasing. This shows that the expected payment is non-decreasing and therefore the expected utility of over-bidding (valuation minus expected payment) can not be greater than the expected utility of bidding truthfully. And as no rational player chooses to play a strategy with worse expected utility, there are no over-bidding strategies in SMCA for non-decreasing, core-selecting payments which satisfy voluntary participation. □

The conditions of corollary 4.4 imply that given any bid-profile b, it is never beneficial to over-bid for any single one of the bidders. Let us again consider bidder i and fixed bids of the other bidders $b_{-i}, b = (b_i, b_{-i})$. A bid b_i' which maximized the utility for bidder i given b_{-i} is called best response for bidder i, $\max_{b_i'} u_i(b_i', b_{-i})$. We name any strategy which increases the utility (but is not necessarily maximal) a better response. Over-bidding is never a better response strategy, in fact we showed that it is dominated by the truthful strategy. That is to say, for any bids of other players b_{-i}, the utility of a truthful strategy is never lesser than the utility of an over-bidding strategy, $u_i(v_i, b_{-i}) \geq u_i(b_i^o, b_{-i})$, for $v_i \leq b_i^o$.

A *Nash equilibrium* is a stable state of a game, in the sense that none of the players have an incentive to change their respective strategies. Any single player deviating their strategy from the Nash equilibrium can not increase their own utility in doing so. Nash equilibria are studied in Game theory to estimate the states to which a game may converge after repeated playing. For instances of over-bidding, one important question is whether over-bidding exists in Nash-equilibria, or if there is always a more profitable option (which excludes over-bidding as a best-response strategy). As we have shown that the truthful bidding strategy dominates any over-bidding strategy we get the following corollary for free.

Corollary 4.5. *Given a CA mechanism $\mathcal{M} = (X, P)$, where X is an allocation algorithm maximizing reported social welfare, and P is a core-selecting, non-decreasing payment satisfying voluntary participation. For single minded bidders, over-bidding strategies can not be part of a Nash equilibria.*

4.4 Proxy and proportional payments

In this section we confirm that the proxy and proportional payment functions are non-decreasing payment rules, taking into account how over-bidding may affect the core.

Claim 4.6. The proxy payment function and the proportional payment function are non-decreasing for single minded CA.

For a fixed allocation x, it has been shown in the paper Bosshard18 [4] that the proxy and the proportional payments are non-decreasing when the core constraints are unaffected by the over-bid. Hence there exists no over-bidding strategy for non-decreasing payment functions, when the over-bid does not affect the core constraints and the allocations x are identical for the over-bid and the truthful bid. We extend the proof to include the cases where over-bidding affects the core and show the proxy and proportional payment are non-decreasing.

The proof of theorem 4.2 shows that any core-selecting payment function satisfying voluntary participation results in payment equal to the over-bid, when over-bidding a losing bid with the minimal winning over-bid. Hence the payment is non-decreasing for such over-bids. Knowing so, we are free to assume that for player i the truthful bid profile $b^v = (v_i, b_{-i})$ is winning, and show that any over-bid from this point on equally results in non-decreasing payment. Combined we then conclude that, independent of the true bid being losing or winning, and independent of the over-bid value, the proxy and proportional payments are non-decreasing.

To reason about the core constraints, we require the notion of a limiting constraint and of relaxing a constraint.

- A *limiting constraint* (or blocking constraint) is an inequality constraint for which equality holds. For core constraints when the lower bound set by a constraint has been reached – in the sense that the minimization problem[2] on the payment has no degree of freedom to further decrease – it is a limiting constraint.
- A *constraint is relaxed* if its bound is modified to increase the range of values the constrained variable(s) can take. On the other hand, a *constraint is strengthened* if a change in the bound reduces the range of values the constrained variable(s) can take. To illustrate, we say a lower bound constraint l on the variable a, $a \geq l$ is relaxed when the right hand side (RHS) of the inequality decreases. Let $l' < l$. Then exchanging l with l' relaxes

[2] The minimization problem refers to the objective function of the proxy, proportional or some other non-decreasing payment. For example the objective function for the proxy payment is the minimal euclidean distance to the origin $\min(\sum_{j \in N} p_j^2)$ under the constraints set by the core and voluntary participation.

the constraint to be $a \geq l'$. Analogously the bound would be strengthened if $l' > l$.

Proof. We prove claim 4.6. It suffices to prove that the changes to the core constraints induced by over-bidding winning bids, can not lead to a decrease in payment for the over-bidder i. There are two conceivable ways how changes to the core constraints can reduce the payment of the over-bidding player i.

1. A blocking constraint on player i is relaxed:
   Then it could be that the payment on i is decreased up to the point where this or another constraint is limiting again.
2. A blocking constraint on player $j \neq i$ is strengthened:
   Allocating a higher payment to player j could lead to reduced payment for player i.

We inspect the ways that over-bidding the i-th player can affect the core. For every possible coalition $L \subseteq N$ there is one constraint:

$$\sum_{j \in N \setminus L} p_j(b, x) \geq \underbrace{W(b_L, X_L(b_L))}_{A} - \underbrace{W(b_L, x_L)}_{B}$$

Any core constraint where $i \notin L$ is unaffected by the over-bid when the allocation x is fixed. Term A has identical value independent of the choice of the winner determination $x, x' \in X(b^v) \cap X(b^o)$. Term B on the other hand is directly dependent on x. While any efficient allocation x, x' has identical social welfare over all N bidders, for different allocations we do not know how the respective terms compare $x' \neq x \implies W(b_L, x_L) \gtreqless W(b_L, x'_L)$. That is why we require a fixed allocation x. For fixed allocation x the term B is unchanged whenever $i \notin L$. Thus we observed that all core constraints on the set $N \setminus L$ where $i \notin L$ are unaffected by over-bidding. This eliminates the possibility to relax a blocking constraint on player i.

Now we list every way a core constraint where $i \in L$ can be affected by an over-bid of player i. Is it possible to strengthen another player's constraint?

Consider the constraint on $N \setminus L,\ i \in L$. For $b^l = (b^l_i, b_{-i})$ and $b^r = (b^r_i, b_{-i})$ where $b^r_i - b^l_i = \epsilon$, for an arbitrarily small $\epsilon > 0$. Since we are considering over-bids on winning bids, b^l_i, b^r_i are winning bids, and thus included in the term B. We list the four cases w.r.t allocations in term A, writing $X_{Lj}(b)$ as a maximal social welfare allocation of the bids in L at index j with bids b:

1. $X_L(b^l)_i = 0 \wedge X_L(b^r)_i = 0$: over-bidding affects the core constraint as sum A is unchanged, but sum B increased by ϵ thus relaxing the constraint by ϵ.

2. $X_L(b^l)_i = 0 \wedge X_L(b^r)_i = 1$: Over-bidding affects the core constraint up to the threshold value b_i^t, $b^t = (b_i^t, b_{-i})$. The threshold value b_i^t is the minimal bid s.t. $W(b^l, X(b^l)) = W(b^t)x^i(b^t))$. Meaning that it is the minimal over-bid by player i s.t. in L the social welfare where b_i^t is a winning bid is equal to the social welfare under any maximal allocation. For some arbitrarily small $\epsilon > 0$, we split this case into instances of cases 1 and 4 as follows: $(X_{Li}(b^l) = 0 \wedge X_{Li}(b_i^t - \epsilon, b_{-i}) = 0), (X_L(b^t)_i = 1 \wedge X_L(b^r)_i = 1)$.

3. $X_L(b^l)_i = 1 \wedge X_L(b^r)_i = 0$: This case can not occur. Winner determination would necessarily allocate the over-bid b^r in L if a lower bid b^l was already winning.

4. $X_L(b^l)_i = 1 \wedge X_L(b^r)_i = 1$: This case leads to no change in the constraint, as the over-bid is represented in both A, B cancelling out.

For a fixed x over-bidding can not cause any core constraint to be strengthened. The only way constraints may be affected by an over-bid, is that some constraints on payments (not including the over-bidder) may relax. Thus the effect over-bidding has on the core-constraints cannot reduce the payment of the over-bidding player i.

This confirms that the proxy and the proportional payments are non-decreasing in single minded CA even when the over-bid affects the core. □

Note that theorem 4.3 does not apply for the VCG-nearest payment. Bosshard18 [4] showed that the VCGN payment is not non-decreasing. The VCG point changes depending on the bids. This change of the optimization target can lead to decreasing payments for over-bids.

Chapter 5

Combinatorial auction classes

As the VCG-nearest payment is not non-decreasing in SMCA, we are interested to find a minimal example of over-bidding for said payment. We want to learn how to distinguish CA models that behave differently - or vice versa, group those auctions which behave equivalently to one another. In doing so, we are able to look for over-bidding examples in increasingly complex models of CA.

A valid solution for winner determination never sells an item more than once and has every bidder win either none or one bundle. For now we ignore the goal to maximize social welfare. We can abstract away the exact contents of the bundles by keeping track of which bundles intersect. This let us model the bids as a graph where V is the set of bids. An edge connects two vertices if the respective bundles do intersect. For single minded CA there's one bid per participant $|V| = |N|$. A valid allocation is then any independent set in the corresponding graph. One can see this as every bidder placed one bid on a single bundle (represented by a vertex). An independent set is a then a set of bundles which do not share any items. Using this model we can list all different auction classes for some $|N|$ by listing all non isomorphic graphs with $|N|$ or less vertices. Figure 5.1 shows all non isomorphic graphs on 2 or 3 vertices.

The *LLG* model [8] (local, local, global) describes a single minded CA with three bidders bidding on two items. The two local bidders want to acquire a single item each, and desire a different item from one another. The Global bidder places a bid on the (global) bundle containing both items.

The LLG model behaves identically when one substitutes the two items by any two independent bundles. It can be represented by the class of single minded CA's with corresponding graph P_3. The two leafs represent the local bidders and do not share an edge. The global vertex however, bids on a bundle which intersects with each of the local bidders bundles.

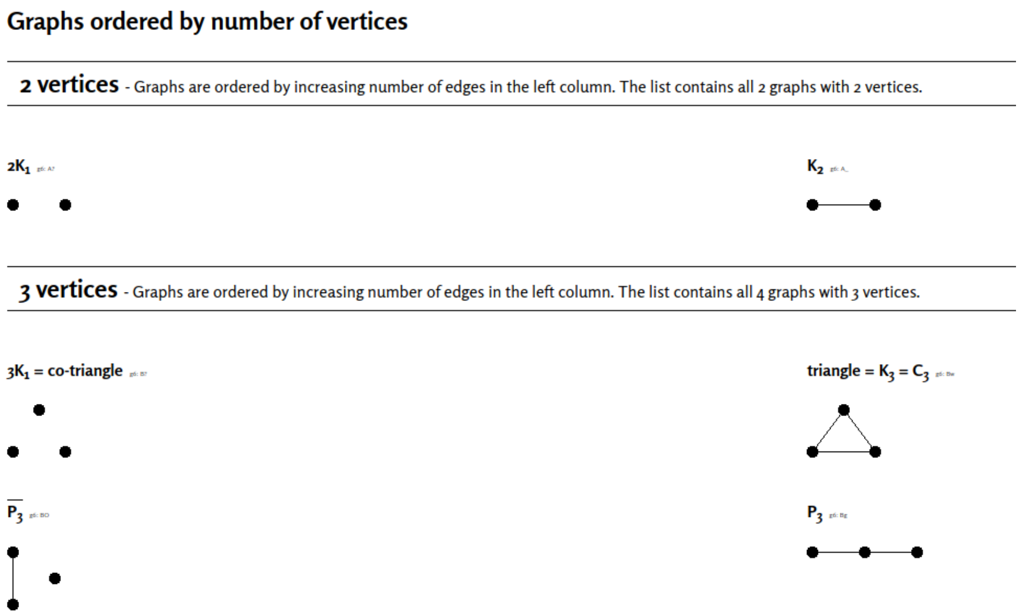

Figure 5.1: List of non isomorphic graphs on 2 or 3 vertices, source: https://www.graphclasses.org/smallgraphs.

5.1 General observations

Before studying specific models let us note some general statements we can make for single minded CA's independent of the number of players taking part.

5.1.1 Independent bids

We say two bids are independent if their bundles are disjoint. For a set T of bids, the bids in T are fully independent if $\forall\ i,\ j \in T$ s.t. $i \neq j$, the respective bundles of i and j are disjoint (do not intersect). In our graph representation this is equivalent to a graph on $|T|$ vertices without any edge.
Clearly the winner determination results in every bid being a winning bid (given that the bid values are > 0). For any subset $L \subseteq N$ we have that, the maximal reported social welfare in the subset L is equal to the reported social welfare of the bids in L under the maximal allocation x over N: $W(b_L, X_L(b_L)) = W(b_L, x_L)$. This corresponds to the RHS of the core constraints, which can then be summarized as $\forall i \in N : p_i(b, x) \geq 0$. For any bid profile b the VCG-nearest, proxy and proportional payments will be 0. There cannot be any over-bidding strategies for independent bids under one of those core-selecting payments.

For disconnected graphs, every connected component behaves like an independent auction to the other components. Between two components there is no overlap in the bundles (and therefore items) that are bid on. Say there are n components. We number the components from 1 to n. One could split the offered items in n sets $(M_1, ..., M_n)$. For every component $i \in [n]$, we do so by including all items that are bid on within the i-th component into set M_i. We disregard items which are never bid on. For every component i, the vertices of the i-th component form sets of bidders $(N_1, ..., N_n)$. Now we interpret every connected component i as a single independent auction of items M_i and with bidders N_i. As every component of a disconnected graph behaves like an independent auction, it suffices to consider connected classes to cover all different CAs. As an example we can view the graph $\bar{P}_3$ (figure 5.1) as $(K_2 \cup K_1)$ or in similar fashion $3K_1$ as $(K_1 \cup K_1 \cup K_1)$.

5.1.2 Fully dependent bids

For a set T of bids, the bids in T are fully dependent if $\forall i, j \in T$ s.t. $i \neq j$, the respective bundles of i and j do intersect. This is represented by complete graph on $|T|$ vertices.
The winner determination must assign the highest bidder of the fully dependent bids the bundle they bid and nothing to the other bidders. The core constraints are such that the winning player must pay at least as much as any of the other bidders is willing to pay. This makes CA with minimal core-selecting payment

functions (VCG-nearest, proxy, proportional) equivalent to a second price auction. As we know second price auctions are strategyproof [15] (the truthful strategy weakly dominates all other strategies). Therefore there are no over-bidding strategies in CA's with fully dependent bids.

5.2 Over-bidding with three or less bidders

With the notion of the CA classes, we began the search for an example of over-bidding in CA for the VCG-nearest payment. Specifically we looked for an example with minimal number of participants. At this point the minimal example we were aware of (table 2.2) contains eleven bidders. Using the graph model we can list all connected non-isomorphic graphs with increasing number of vertices and begin to examine them. Every graph on three or less vertices except the path on 3 vertices P_3 consists of fully connected and or fully independent components. Thus, as argued in the general observations section 5.1, we know that there exist no over-bidding strategies in those classes. We examined P_3 as generalized LLG model. Another way to see P_3 is as a 2-star graph, where the center vertex (G) is connected to each of the other two vertices (L). Based on the explicit formula of the VCG-Nearest payment as characterized in the paper [11], it is clear that the VCGN payment is non-decreasing in the P_3 class. By corollary 4.4 we know that, for non-decreasing, core-selecting payments satisfying voluntary participation, there are no over-bidding strategies in SMCA. This concludes that there are no over-bidding strategies for the VCGN payment in SMCA with less then four bidders.

5.2.1 The link to single constraint classes

As a first instance of connected graphs on four vertices we studied the 3-star graph. It represents the class equivalent to the LLLG model, a simple extension of the LLG model by adding another local (independent) bundle and bidder (L) and where the global bidder is again interested in all items. To examine it we derived an explicit formulation of the VCGN payment for this model and then showed that it is non-decreasing. From this evolved the idea to study the n-star class we name L*G, instead of brute-forcing the various classes on four vertices. The L*G class characterizes the class with n local bidders and one global bidder. The analysis of L*G then lead to the result on SMCAs with a single effective core constraint (SECC). In the next chapter we first show the result on SECC. Then we present L*G and apply said result. Since LLG and LLLG are instances of L*G, this result covers those two individual cases, hence we refrain from writing down the individual proofs.

Chapter 6

single effective core constraint

For core selecting payments, the core constraints bound the payments. If one lists every core constraint on the set of winning bidders, we often observe that many of the constraints are obsolete in the presence of a more restrictive constraint. For example if we look at the following three inequalities we observe that the last constraint encompasses the first two.

$$p_1 \geq b_G - b_2$$
$$p_2 \geq b_G - b_1$$
$$p_1 + p_2 \geq b_G$$

Since voluntary participation implies that $b_i \geq p_i$ and for $b_G \geq 0$, when the constraint $p_1 + p_2 \geq b_G$ is satisfied so are the others. We say a constraint α *covers* constraint β if $\alpha \implies \beta$. For $b_1 + b_2 \geq b_G$ and $b_G \geq b_i$ for $i \in \{1, 2\}$ those are the core constraints of the LLG class. For specific payment functions like the VCG-Nearest payment, we can further ignore all constraints which are implicitly satisfied by the payment. Notably those are the constraints on a single payment for which the right hand side (RHS) is equivalent to the VCG payment. Those are implicitly satisfied, as $p_{VCGN_i} \geq p_i^V$, which states that at every position i, VCGN payment is never lesser than VCG payment.

Definition 6.1. A CA class has a **single effective core constraint** (SECC) if the core constraints can be fully expressed by a single constraint. We call such classes single core constraint classes.

As we argued above, the core constraints of LLG can be reduced to a single effective core constraint, namely $p_1 + p_2 \geq b_G$.

Theorem 6.2. *For SMCA classes where the core constraints can be represented by a single effective core constraint, the VCGN payment is non-decreasing and therefore there are no over-bidding strategies.*

Let $n \geq 1$ be the number of winning bidders. We will show theorem 6.2 by deriving the representation $p_{VCGN} = p^V + x$ for some x, where for all

$i \in [n]$, $x_i \geq 0$. Then we provide the optimal point $\mathbf{x}^*$ which added to the VCG point p^V results in the VCG-Nearest point. Finally, from this explicit formulation of the VCG-Nearest point, we deduce that it is non-decreasing in single core constraint classes.

6.1 Explicit VCGN formulation

This section derives an explicit formulation of the VCGN-point for SMCA classes whith a single effective core constraint.

6.1.1 Shift from VCG point to origin

For the set of winning bidders L, we formulate the single core constraint as follows: $\sum_{i \in L} p_i(b, x) \geq B_R$. This core constraint covers all other core constraints, i.e. when $\sum_{i \in L} p_i(b, x) \geq B_R$ holds, every core constraint is satisfied. We compare the sum of winning player's VCG-payments to the RHS value B_R of the single core constraint.

Lemma 6.3. *Let L be the set of winning bidders, $|L| = n$, $\sum_{i=1}^n b_i = B_L \geq B_R$. We show that $\sum_{i=1}^n p_i^V \leq B_R$.*

Proof. For a SMCA with singe effective core constraint $\sum_{i \in L} p_i(b, x) \geq B_R$. We assume $\sum_{i=1}^n p_i^V > B_R$ and do a proof by contradiction. The core constraint on any single winning payment p_l, for $l \in L$ is given by

$$p_l \geq W(b_{-l}, X_{-l}(b_{-l})) - W(b_{-l}, x_{-l}) = p_l^V$$

$$\text{for } L \text{ being the set of winning bids, let } B'_R = \sum_{i=1}^n p_i^V = \sum_{l \in L}^n p_l^V \leq \sum_{l \in L}^n p_l$$

The assumption $B_R < \sum_{i=1}^n p_i^V$ implies that not all the core constraints on the single payments, are covered by the single effective constraint. Summing up the constraints on single payments $\sum_{l \in L}^n p_l^V = B'_R > B_R$ forms a constraint which is not covered by the constraint characterized by B_R. This is a contradiction to the assumption that B_R is a single effective core constraint, which requires that it covers all other core constraints. □

Using the result above, in SMCA classes with a single effective core constraint we can represent the VCGN payment of any player i as $p_{VCGN_i} = p_i^V + x_i$ for some $x_i \geq 0$. Let us adapt the constraint such that we can look for the closest point $\mathbf{x}^*$ on the modified constraint to the origin instead of the vcg-point. For a single constraint, the total VCGN payments are always equal to the RHS value of the constraint B_R. It is necessarily a blocking constraint, making it an equality constraint.

$$|p_{VCGN}|_1 = \sum_{i \in L} p_i^V + x_i = B_R \iff \sum_{i \in L} x_i = B_R - \sum_{i \in L} p_i^V = |\mathbf{x}^*|_1$$

6.1.2 Intuitive deduction of explicit formula

With a single blocking constraint, the difference of the total payment and the fixed allocated VCG-payments $\Delta = \sum_{i \in L} x_i = B_R - \sum_{i \in L} p_i^V$, with $\Delta \geq 0$ by Lemma 6.3. We need to find the x_i that minimize the distance to the core while maintaining voluntary participation $p_i \leq b_i \implies x_i \leq b_i - p_i^V$. We denote the constraints resulting from voluntary participation as $b_i' = b_i - p_i^V$. A constraint b_i' can be seen as that players budget, since after paying the VCG-payment player i is willing to pay up to the budget b_i' to acquire corresponding bundle to the bid. W.l.o.g. we say that the constraints b_i' are sorted increasingly. I.e. the lowest index is bound by the smallest value: $b_1' = b_1 - p_1^V \leq b_2' \leq \cdots \leq b_n' = b_n - p_n^V$. The core constraint $\sum_i^n x_i = \Delta$ can be seen as the hyperplane in $\mathbb{R}_+^n$ defined by $(\sum_i^n x_i) - \Delta = 0$ and has the n-dimensional normal vector $\vec{n} = (1, \ldots, 1)^\top$. Hence the closest point $p^\star$ on the hyperplane to the origin lies in direction of $\vec{n}$ at euclidean distance $\frac{\Delta}{\sqrt{n}}$, $p^\star = \frac{\Delta}{n}\vec{n}$. If no budget constraint is violated by $p^\star$, then $b_1 - p_1^V \geq \frac{\Delta}{n}$, and $p^\star$ represents p_{VCGN}. This would translate to $x_i = \frac{\Delta}{n}$, for every i. Else the closest point $\mathbf{x}^*$ is constrained by some budget limit in (at least) the first of the sorted indices $\mathbf{x}_1^* = b_1' < \frac{\Delta}{n}$. Let Δ_j be the remainder of the total payment (s.t. it sums up to Δ), to be assigned among winning bidders with index greater j, when the j payments with least indices reached their respective budget limits ($\forall i \in [j]$, $\mathbf{x}_i^* = b_i'$)

$$\Delta_j = \begin{cases} \Delta & \text{if j} = 0 \\ \Delta - \sum_{i=1}^{j} b_i' & \text{for } 1 \leq j \leq n \end{cases}$$

For $0 \leq k \leq n-1$ let k be the smallest index s.t. we can distribute Δ_k evenly among the $n-k$ players with greatest budgets (the greatest $n-k$ indices) while respecting the constraints b_i' for every winning bidder i. Such a k always exists for winning players as $\sum_{i \in L} b_i > B_R \implies \sum_{i \in L}(b_i - p_i^V) > B_R - \sum_{i \in L} p_i^V = \Delta$. Thus, $\frac{\Delta_k}{n-k} \leq b_{k+1}'$ and for $k \geq 1$, $\frac{\Delta_{k-1}}{n-k+1} > b_k'$. We claim that the closest point to the origin lying in the core[1] is $\mathbf{x}^*$.

Claim 6.4.

$$\mathbf{x}_i^* = \begin{cases} b_i' = b_i - p_i^V & \text{if } 1 \leq i \leq k \\ \frac{\Delta_k}{n-k} & \text{if } k+1 \leq i \leq n \end{cases}$$

When $k = 0$ we get an even distribution of the remaining payment, $\frac{\Delta_k}{n-k}$ for all $i \in [n]$.

[1]Here the single core constraint is $\sum_{i \in L} x_i = \Delta$.

6.1.3 Optimality certificate for $\mathbf{x}^*$

To calculate point $\mathbf{x}^*$ we employ a quadratic program. According to the CGAL definition [16] a quadratic program (QP) has the following form:

$$\begin{aligned} \text{minimize } \mathbf{x}^T D\mathbf{x} + \mathbf{c}^T + c_0 \\ \text{subject to } A\mathbf{x} \gtreqless b \\ \mathbf{l} \leq \mathbf{x} \leq \mathbf{u} \end{aligned}$$

in n real variables $\mathbf{x} = (x_0, \ldots, x_{n-1})$.

- A is an $m \times n$ constraint matrix
- $\mathbf{b}$ is an m dimensional right-hand side vector
- $\gtreqless$ is an m-dimensional vector of relatrions from $\{\leq, =, \geq\}$
- D is a symmetric positive-semidefinite $n \times n$ matrix (the quadratic objective function)
- n-dimensional vector of lower bounds $\mathbf{l}$ and upper bounds $\mathbf{u}$ for $\mathbf{x}$
- n-dimensional $\mathbf{c}$, the linear objective function
- c_0 a constant of the objective function

We aim to show claim 6.4. To do so we first declare the quadratic problem, then provide an optimality certificate λ, and finally show that λ satisfies the conditions of lemma 6.5. The lemma 6.5 and its proof are taken from the CGAL documentation [17].

Lemma 6.5 (optimality certificate)**.** *A feasible n-vector $\mathbf{x}^*$ is an optimal solution of the Quadratic Program if an m-vector λ with the following properties exists.*

1. *if the i-th constraint is of type $\leq$ ($\geq$, respectively), then $\lambda_i \geq 0$ ($\lambda_i \leq 0$, respectively).*
2. $\lambda^T(A\mathbf{x}^* - \mathbf{b}) = 0$
3. $(c^T + \lambda^T A + 2\mathbf{x}^{*T} D)_j \begin{cases} \geq 0, & \text{if } \mathbf{x}^*_\mathbf{j} = l_j < u_j \\ = 0, & \text{if } l_j < \mathbf{x}^*_\mathbf{j} < u_j \\ \leq 0, & \text{if } l_j < u_j = \mathbf{x}^*_\mathbf{j} \end{cases}$

The proof of lemma 6.5 follows.

Proof. Let $\mathbf{x}$ be any feasible solution. We need to prove that

$$\mathbf{c}^T\mathbf{x} + \mathbf{x}^T D\mathbf{x} \geq \mathbf{c}^T\mathbf{x}^* + \mathbf{x}^{*T} D\mathbf{x}^*$$

For this, we argue as follows.

$$\begin{aligned}
\mathbf{c}^T\mathbf{x} + 2\mathbf{x}^{*T}D\mathbf{x} &\geq \mathbf{c}^T\mathbf{x} + 2\mathbf{x}^{*T}D\mathbf{x} + \lambda^T(A\mathbf{x} - \mathbf{b}) && \text{(by } A\mathbf{x} \gtreqless b \text{ and 1.)}\\
&= (\mathbf{c}^T + \lambda^T A + 2\mathbf{x}^{*T}D\mathbf{x})\mathbf{x} - \lambda^T b\\
&\geq (\mathbf{c}^T + \lambda^T A + 2\mathbf{x}^{*T}D\mathbf{x})\mathbf{x}^* - \lambda^T b && \text{(by } \mathbf{l} \leq \mathbf{x} \leq \mathbf{u} \text{ and 3.)}\\
&= \mathbf{c}^T\mathbf{x}^* + 2\mathbf{x}^{*T}D\mathbf{x}^* && \text{(by 2.)}
\end{aligned}$$

After adding $\mathbf{x}^T D\mathbf{x} - \mathbf{x}^T D\mathbf{x} + \mathbf{x}^{*T} D\mathbf{x}^* = \mathbf{x}^{*T} D\mathbf{x}^*$ to both sides of this inequality, we get

$$\mathbf{c}^T\mathbf{x} + \mathbf{x}^T D\mathbf{x} - (\mathbf{x} - \mathbf{x}^*)^T D(\mathbf{x} - \mathbf{x}^*) \geq \mathbf{c}^T\mathbf{x}^* + \mathbf{x}^{*T} D\mathbf{x}^*,$$

and since D is positive semidefinite, we have $(\mathbf{x} - \mathbf{x}^*)^T D(\mathbf{x} - \mathbf{x}^*) \geq 0$ and the lemma follows. □

With this we are ready to prove claim 6.4: $\mathbf{x}_i^* = \begin{cases} b_i' = b_i - p_i^V & \text{if } 1 \leq i \leq k \\ \frac{\Delta_k}{n-k} & \text{if } k+1 \leq i \leq n \end{cases}$

Proof. We formulate the quadratic problem, which solves for the minimal (squared) distance of a point within the core to the origin with respect to the constraints, as follows:

$$\begin{aligned}
\text{minimize } & \mathbf{x}^T\mathbf{x}\\
\text{subject to } & A\mathbf{x} \geq b\\
& l \leq x
\end{aligned}$$

Here we have

- n real variables $\mathbf{x}$ representing the payment
- A is an $m \times n$ constraint matrix, where $m = n+1$, for n budget constraints and one core constraint.
- $\mathbf{b}$ is an m dimensional right-hand side vector
- $\mathbf{l}$ lower-bounds $\mathbf{x}$ s.t. $\forall i \in [n]: \; x_i \geq 0$
- D the $n \times n$ identity matrix

$$A\mathbf{x} \geq \mathbf{b} \iff \begin{bmatrix} -1 & & \\ & \ddots & \\ & & -1 \\ 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \geq \begin{bmatrix} -b_1' \\ \vdots \\ -b_n' \\ \Delta \end{bmatrix} \tag{6.1}$$

Thus, for $1 \leq j \leq n+1$ we have $(A\mathbf{x} \geq \mathbf{b})_j = \begin{cases} -x_{j-n} \geq -b_{j-n}' & \text{if } 1 \leq j \leq n \\ \sum_{i=1}^n x_i \geq \Delta & \text{if } j = n+1 \end{cases}$

We show that the optimal point is $\mathbf{x}^*$ by providing a certificate λ which satisfies the conditions of Lemma 6.5.
As reference, when $k > 1$, $\quad \mathbf{x}^{*T} = \begin{bmatrix} b'_1 & \dots & b'_k & \frac{\Delta_k}{n-k} & \dots & \frac{\Delta_k}{n-k} \end{bmatrix}$.
We claim that the following λ is an optimality certificate for $\mathbf{x}^*$

$$\text{For } 1 \leq j \leq n+1, \quad \lambda_j = \begin{cases} 2b'_j - \frac{2\Delta_k}{n-k} & \text{if } 1 \leq j \leq k \\ 0 & \text{if } k+1 \leq j \leq n \\ -\frac{2\Delta_k}{n-k} & \text{if } j = n+1 \end{cases} \tag{6.2}$$

Hence for $k > 1$, $\lambda^T = \begin{bmatrix} (2b'_1 - \frac{2\Delta_k}{n-k}) & \dots & (2b'_k - \frac{2\Delta_k}{n-k}) & 0 & \dots & 0 & -\frac{2\Delta_k}{n-k} \end{bmatrix}$
We show the conditions of the Lemma 6.5 in order.

1. Since all constraint relations are $\geq$[2], we require $\lambda_i \leq 0$, for all $i \in [n+1]$. From the number of winning players n follows $n > k$. The definition of Δ_k entails that $k \geq j \implies 0 \leq b'_j < \frac{\Delta_k}{n-k}$. Therefore, by definition of λ (equation 6.2) at every index j, $1 \leq j \leq n+1$, the value of λ is $\lambda_j \leq 0$.

2. Show that $\lambda^T(A\mathbf{x}^* - \mathbf{b}) = 0$:
   We have $(A\mathbf{x}^* - \mathbf{b})_j = \begin{cases} 0 & \text{if } 1 \leq j \leq k \\ b'_j - \frac{\Delta_k}{n-k} & \text{if } k+1 \leq j \leq n \\ 0 & \text{if } j = n+1 \end{cases}$
   It is easy to see that $\lambda^T(A\mathbf{x}^* - \mathbf{b}) = 0$, as for every $j \in [n+1]$ either $\lambda_j = 0$ or $(A\mathbf{x}^* - \mathbf{b})_j = 0$.

3. We show that for $l_j \leq \mathbf{x}^*_\mathbf{j} \leq u_j : \quad (c^T + \lambda^T A + 2\mathbf{x}^{*T} D)_j = 0$:
   For our quadratic program the expression simplifies to $(\lambda^T A + 2\mathbf{x}^{*T})_j$ as there is no linear objective and the matrix D is the identity matrix.

   $$(\lambda^T A)_j = \begin{cases} -2b'_j & \text{if } 1 \leq j \leq k \\ -2\frac{\Delta_k}{n-k} & \text{if } k+1 \leq j \leq n \end{cases} \quad 2(\mathbf{x}^{*^T})_j = \begin{cases} 2b'_i & \text{if } 1 \leq j \leq k \\ 2\frac{\Delta_k}{n-k} & \text{if } k+1 \leq j \leq n \end{cases}$$

   This concludes that for $j \in [n] : \quad (\lambda^T A + 2\mathbf{x}^{*T})_j = 0$

□

Having proven claim 6.4 it follows that $p_{VCGN_i} = \begin{cases} b_i & \text{if } 1 \leq i \leq k \\ \frac{\Delta_k}{n-k} + p_i^V & \text{otherwise} \end{cases}$
We get this by adding $\mathbf{x}^*$ to the VCG-point.

[2] The core constraint can be formulated with $\geq$ instead of equality, as the objective function will be minimized until the core constraint is blocking and equality holds.

6.2 Analyzing over-bidding in SMCA with SECC

We can deduct from the above formulation of the VCG-Nearest payment that it is non-decreasing in classes with a single effective core constraint. This then implies that there are no over-bidding strategies in SMCA with SECC using VCG-Nearest payment.

Consider winning bidder i, truthful bid v_i and over-bid b_i^o. For some $\epsilon > 0$ we have $v_i + \epsilon = b_i^o$. Bids of the other bidders b_{-i} are fixed. All participants bids are denoted $b^v = (v_i, b_{-i})$ and $b^o = (b_i^o, b_{-i})$ respectively. Let us look at the budget $b'_j = b_j - p_j^V(b)$ of the winning players $j \in [n]$. The budget b'_j represents the upper bounds on payment x_j in the QP after we shift the objective to minimize the distance from VCG to the origin. We will distinguish the budget w.r.t. player i over-bidding by noting $b'^{v}_j = b_j^v - p_j^V(b^v)$ and $b'^{o}_j = b_j^o - p_j^V(b^o)$. The truthful bids b^v are sorted in ascending order, for $l, j \in [n],\ l < j,\ b_l^v \leq b_j^v$. Arrange the over-bid vector b^o to contain the same players bid at the same position as the ordering of the truthful bids. Then for all indices $j \in [n],\ j \neq i$ we have $b_j^o = b_j^v$. Solely the over-bid at position i differs, which may cause b_j^o to no longer being sorted.

When we insert the definition of the VCG-payment (and under the assumption that $\sum_{j \in L} b_j = B_L > B_R$) the budget reads:

$$b'_j = \begin{cases} b_j & \text{if } p_j^V = 0 \iff b_j < B_L - B_R \\ B_L - B_R & \text{otherwise} \implies p_j^V > 0 \end{cases}$$

Let s be the number of bids from winning players which have VCG-payment of zero. For $s > 0$ this implies $b_s < B_L - B_R$. All indices greater than s result in the same budget since $b'_j = b_j - p_j^V = b_j - \left(B_R - \sum_{l \neq j \wedge l \in L} b_l\right) = \left(\sum_{l \in L} b_l\right) - B_R$. An ascending (or descending) ordering of the bids b is maintained among the corresponding budgets b'. As we know that there exists a minimal $k,\quad 0 \leq k \leq n-1$ s.t. $\frac{\Delta_k}{n-k} \leq b'_{k+1}$, it follows for the greatest budget $b'_n \geq \frac{\Delta_k}{n-k}$, or else no solution would be possible[3]. Additionally, the greatest budget b'_n cannot exceed $B_L - B_R$.

Compare the VCG-payments of the truthful bid and the over-bid. The over-bidder i has identical VCG-payment $p_i^V(b^v) = p_i^V(b^o)$ as all other bids are fixed. The VCG-payment of any other bidder may decrease by up to $\epsilon = b_i^o - v_i$ (but not increase). The sum of payments Δ which are not fixed by the VCG payments may increase as result of over-bidding. We introduce Δ^v and Δ^o, the difference of payments from the core constraint B_R to the sum of VCG payments

[3]We know that there must always be a solution due to $B_L \geq B_R$.

in dependence of player i over-bidding.

$$\text{for } j \in L,\ j \neq i,\ p_j^V(b) = \max\big(0, B_R - \sum_{l \in L \wedge l \neq j} b_l\big)$$

$$p_j^V(b^v) \geq p_j^V(b^o) \implies B_R - \sum_{l \in L} p_l^V(b^v) = \Delta^v \leq \Delta^o = B_R - \sum_{l \in L} p_l^V(b^o)$$

Let $\Delta_j^v = \Delta^v - \sum_{l=1}^{j} \big(b_l^v - p_l^V(b^v)\big)$. For $\Delta_j^v, 1 \leq k \leq n$ let k be the smallest index s.t. we can distribute the remaining Δ_k^v payment evenly among the $n-k$ players with greatest budgets $b'^v = b^v - p^V(b^v)$. By analogy, we define $\Delta_j^o = \Delta^o - \sum_{l=1}^{j} \big(b_l^o - p_l^V(b^o)\big)$. And for $\Delta_j^o, \quad 1 \leq k' \leq n$ let k' be the smallest index s.t. we can distribute the remaining $\Delta_{k'}^o$ payment evenly among the $n-k'$ players with greatest upper bounds $b'^o = b^o - p^V(b^o)$. For every bidder j, the budgets ${b'_j}^v$ for bids b_j^v and ${b'_j}^o$ for b_j^o compare as follows.

$$p_j^V(b^v) \geq p_j^V(b^o) \implies$$
$${b'_j}^v = b_j - p_j^V(b^v) \leq b_j - p_j^V(b^o) = {b'_j}^o$$

Lemma 6.6. *For some bidder i, we consider truthful bid profile $b^v = (v_i, b_{-i})$ and over-bid profile $b^o = (b_i^o, b_{-i})$, $b_i^o \geq v_i$. Then, for the VCG-Nearest payment and single constraint CA classes, we have*

$$\frac{\Delta_k^v}{n-k} \leq \frac{\Delta_{k'}^o}{n-k'}$$

Remember that we sorted the budgets b'^v in ascending order and arranged b'^o s.t. the every bidder is ordered according to the ordering in b'^v. We prove lemma 6.6 by showing that $k' \geq k$, as we know $\Delta^o \geq \Delta^v$. Here's a concise proof by contradiction.

Proof. Assume $k' < k$. Then by definition we must have $\frac{\Delta_k^o}{n-k} > {b'_{k+1}}^o$ and $\frac{\Delta_k^v}{n-k} \leq {b'_{k+1}}^v$. We have ${b'_{k+1}}^o \geq {b'_{k+1}}^v$, and $\Delta_k^v \leq \Delta_k^o$ both following from the fact that over-bidding can reduce some VCG payments. Then, $\frac{\Delta_k^o}{n-k} > {b'_{k+1}}^o \geq {b'_{k+1}}^v \geq \frac{\Delta_k^v}{n-k}$. This stands in contradiction to $\Delta_k^v \leq \Delta_k^o$. The conclusion is $k' \geq k$. □

Finally, we are ready to prove theorem 6.2. As recapitulation, we have shown the following explicit formulation for the VCGN-payments in SMCA with SECC:

$$p_{VCGN_i} = \begin{cases} b_i & \text{if } 1 \leq i \leq k \\ \frac{\Delta_k}{n-k} + p_i^V & \text{otherwise} \end{cases}$$

Where k is the smallest index $0 \leq k \leq n-1$ s.t. we can distribute Δ_k evenly among the $n-k$ players with greatest budgets (the greatest $n-k$ indices) while respecting the constraints b'_i for every winning bidder i. And Δ_k being

$$\Delta_k = \begin{cases} \Delta & \text{if k} = 0 \\ \Delta - \sum_{i=1}^{j} b'_i & \text{for } 1 \leq k \leq n \end{cases}$$

Proof. To show theorem 6.2, we do a case distinction on the VCG-Nearest payments player i bidding truthfully or over-bidding to show that the payments are non-decreasing.

- $(p^v_{VCGN_i} = v_i) \wedge (p^o_{VCGN_i} = b^o_i)$
  As $b^o_i > v_i$ this is an increase in payment.
- $(p^v_{VCGN_i} = v_i) \wedge (p^o_{VCGN_i} = \frac{\Delta^o_{k'}}{n-k'} + p^V_i)$
  In this case $v_i < \frac{\Delta^v_k}{n-k} + p^V_i$ which by lemma 6.6 $\frac{\Delta^v_k}{n-k} \leq \frac{\Delta^o_{k'}}{n-k'}$ cannot decrease the payment.
- $(p^v_{VCGN_i} = \frac{\Delta^v_k}{n-k} + p^V_i) \wedge (p^o_{VCGN_i} = b^o_i)$
  Here $\frac{\Delta^v_k}{n-k} + p^V_i \leq v_i < b^o_i$, and therefore it is an increase in payment.
- $(p^v_{VCGN_i} = \frac{\Delta^v_k}{n-k} + p^V_i) \wedge (p^o_{VCGN_i} = \frac{\Delta^o_{k'}}{n-k'} + p^V_i)$
  Again by lemma 6.6 $\frac{\Delta^v_k}{n-k} \leq \frac{\Delta^o_{k'}}{n-k'}$ the over-bid cannot result in a lesser payment.

□

To summarize we showed that the VCG-Nearest payment is non-decreasing for single minded combinatorial auction classes with a single effective core constraint. By corollary 4.4 this proves that there exist no over-bidding strategies in SMCA with single effective core constraint.

Corollary 6.7. *SMCA classes where any winner determination results in two or less winning bidders have no over-bidding strategies under the VCGN-payment.*

We prove 6.7 by showing that two or less winners implies a single core constraint. Then it follows from theorem 6.2

Proof. It suffices to look at the core constraints of the (two or less) winners. We know that core constraints for single players are by definition equal to the VCG-payment and therefore implicitly satisfied by VCGN-payment. If there are two winners, the core constraint which lower-bounds the sum of the winning players with the greatest RHS is most limiting. This blocking constraint covers all other (non-trivially satisfied) constraints. The core is fully defined by this most limiting core constraint, which is the single effective core constraint. □

The corollary 6.7 lets eliminate over-bidding strategies under the VCGN-payment in every SMCA class where there are at most two winners. Considering all (non-isomorphic) connected graph's on 4 or less vertices, only the class represented by the 3-star graph allows for an independent set of three or more vertices. Hence only the SMCA class represented by the 3-star graph can have three winners. For all other SMCA classes on 4 or less vertices, corollary 6.7 implies that there exist no over-bidding strategies under VCGN payment. Next section covers the LLLG class indirectly by showing that for any $n \geq 1$, an n-star class is a single core constraint class. This let us apply theorem 6.2 to state that the VCGN payment behaves non-decreasingly for n-star classes, which makes them robust against over-bidding.

6.3 L*G

With L*G we consider all the classes of CA which can be modeled as star graphs. The star graphsfrom three to seven vertices are depicted in figure 6.1. In other words, L*G is the class of all CAs where there is one global bidder and some number of local bidders. The global bidder bids on a bundle which intersects with every other bidders bundle in one or more items. Among the local bidders, the selected bundles are mutually independent. We say there are $n \geq 1$ local bidders who form the set L.

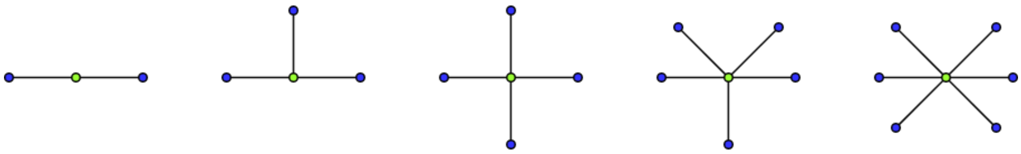

Figure 6.1: The star graphs with 3, 4, 5, 6 and 7 vertices. The central vertex (yellow) represents the global bid, which intersects with every of the local (blue) bids.

We show that L*G is a single core constraint class. This implies that there are no over-bidding strategies for the VCGN payment in single minded CA of the L*G class.

Should the global player win, the payment is given as $p_G = \sum_{i \in L} b_i$. The global players utility is only positive if its valuation is greater than the corresponding payment. Therefore the global bidders dominant strategy is to place a truthful bid b_G. The global player has no over-bidding strategies. As shown in theorem 4.2 there can only be over-bidding in SMCA when a truthful bid would be winning. That is why we consider outcomes of local bidders victory. Note that a sufficient argument against over-bidding by the global player is to show that this case implies a single effective core constraint (theorem 6.2): $p_G \geq \sum_{i \in L} b_i$.

Or even simpler, by corollary 6.7 to argue that in this case there are less than two winners.

6.3.1 Core constraints in L*G

When local players win there is effectively a single core constraint in L*G. Namely on the payments p_i of local players $i \in L$:

$$\sum_{i \in L} p_i \geq b_G = B_R$$

Here the global player's bid directly translates to the best allocation in $N \setminus L = B_R$. To prove this, consider any coalition $C \subseteq L$, the corresponding core constraint is given as:

$$\sum_{i \in L \setminus C} p_i \geq \begin{cases} 0 & \text{if local players in } C \text{ form a better solution than } b_G \\ b_G - \sum_{i \in C} b_i & \text{otherwise} \end{cases}$$

$$\begin{aligned} \text{where } \sum_{i \in L \setminus C} p_i &\geq b_G - \sum_{i \in C} b_i \\ \iff \sum_{i \in L \setminus C} p_i + \sum_{i \in C} b_i &\geq b_G \\ \text{and due to } p_i \leq b_i : \sum_{i \in L \setminus C} p_i + \sum_{i \in C} b_i &\geq \sum_{i \in L} p_i \end{aligned}$$

It follows that the most restraining constraint is for $C = L$, the set of all local players.

$$\sum_{i \in L} p_i \geq b_G$$

Hence, L*G classes have are single effective core constraint, and by theorem 6.2 we know that there are no over-bidding strategies.

CHAPTER 7

Searching for minimal examples of over-bidding

7.1 Listing SCMA classes with five bidders and three winners

Based on corollary 6.7, which states that there are no over-bidding strategies for VCGN payment when there are two or less winners, we continue the search for a minimal example of over-bidding under VCGN payment in SMCA with five bidders. There are nine non-isomorphic connected simple graphs[1] on five vertices that contain one independent sets of size three. They are depicted in figure 7.1. One of them is 5-path, which we will discuss in detail in the next section. Another is the self complementary 'bull', which we can see as 5-path with the additional edge (m_l, m_r). We will also discuss bull in more detail. Then there are seven graphs which are LLLG + 1 connected vertex. We name those 7 cases LLLG+$1_{\{g,l\}}$ for $g \in \{0,1\}, l \in \{0,1,2,3\}$ denoting the number of edges to the global or local players. We require at least one edge between the added vertex (+1) and LLLG to have a connected graph, thus $l + g$ can't be zero. Due to symmetry among the local vertices for $l > 0$, the l edges can connect to any l different vertices among the local ones.

LLLG+$1_{\{1,0\}}$ is equivalent to LLLLG, for which we know that it is a single core constraint class, as it belongs to the classes of star graphs discussed in section 6.3. On a short glance, LLLG+$1_{\{0,3\}}$ looks like it can be reduced to LLLG by considering the added vertex (+1) and the global bidder G to be the a single global bidder G'. With this interpretation we would sum up their bids $b_{G'} = b_G + b_{+1}$ to get the bid of the new global bidder $b_{G'}$, which we can do, as they are independent of one another and can never be allocated together with any local bid. Similarly for LLLG+$1_{\{1,3\}}$ it seems that we can reduce it to LLLG by discarding the smaller bid of the added vertex and the global bidder. The remaining six classes (5-path, bull, LLLG+$1_{\{0,1\}}$, LLLG+$1_{\{1,1\}}$, LLLG+$1_{\{0,2\}}$,

[1] Simple graphs have no loops, or multiple edges or edge weights.

LLLG+1$_{\{1,2\}}$) are interesting candidates to look for over-bidding under VCG-Nearest payment.

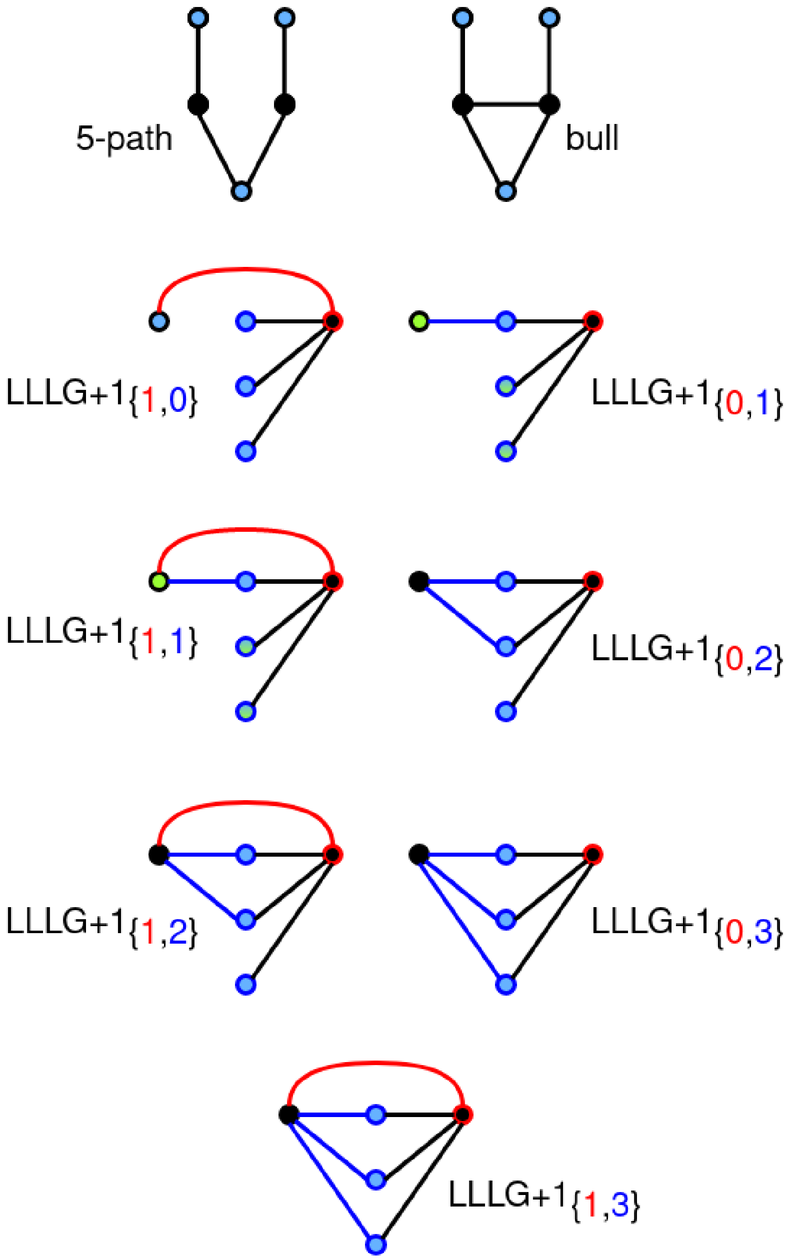

Figure 7.1: All connected SCMA classes on five vertices which allow for three winners (light blue vertices). LLLG+1$_{\{0,1\}}$ and LLLG+1$_{\{1,1\}}$ have two possible allocations of three winners, pairing the two local vertices with lighe blue-yellow gradient with the third local vertex or yellow +1 vertex. LLLG+1$_{\{1,0\}}$ has an allocation of four winners.

7.2 5-path

The 5-path graph represents a CA class that allows outcomes of three winners, namely the leaves l, r and the middle node m. We want to verify if the core constraints can be reduced to a single effective constraint, and if not see if there can exist over-bidding strategies. We name the vertices of the 5-path in this order l, m_l, m, m_r, r, as drawn in figure 7.2.
The valid maximal assignments are $(\{l, m, r\}, \{l, m_r\}, \{m_l, m_r\}, \{m_l, r\})$.

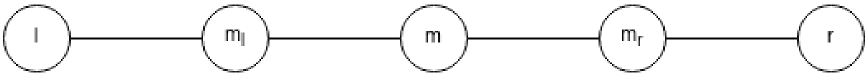

Figure 7.2: Five bidders placing one bid each represent the vertices. Edges indicate a non-empty intersection of items among the connected bundles.

cID	payment variables						$WD(cID) - W(b_{cID}, x_{cID})$
0	p_l	$+$	p_m	$+$	p_r	$\geq$	$b_{m_l} + b_{m_r}$
1	0	$+$	p_m	$+$	p_r	$\geq$	$\max\big((b_{m_l} + b_{m_r}), (b_l + b_{m_r})\big) - b_l$
2	p_l	$+$	0	$+$	p_r	$\geq$	$\max\big((b_{m_l} + b_{m_r}), b_m\big) - b_m$
3	0	$+$	0	$+$	p_r	$\geq$	p_r^V
4	p_l	$+$	p_m	$+$	0	$\geq$	$\max\big((b_{m_l} + b_{m_r}), (b_{m_l} + b_r)\big) - b_r$
5	0	$+$	p_m	$+$	0	$\geq$	p_m^V
6	p_l	$+$	0	$+$	0	$\geq$	p_l^V

Table 7.1: Core constraints of 5-path, given that (l, m, r) is the winning assignment. We use the constraint identifier cID to refer to a specific core constraint.

$$\begin{aligned}
p_r^V &= \max\big((b_{m_l} + b_{m_r}), (b_l + b_m), (b_l + b_{m_r})\big) && -(b_l + b_m) \\
p_m^V &= \max\big((b_l + b_r), (b_l + b_{m_r}), (b_{m_l} + b_{m_r}), (b_{m_l} + b_r)\big) && -(b_l + b_r) \\
p_l^V &= \max\big((b_{m_l} + b_{m_r}), (b_m + b_r), (b_{m_l} + b_r)\big) && -(b_m + b_r)
\end{aligned}$$

7.2.1 Reducing core constraints

We list all core constraints in table 7.1. The cIDs 3,5 and 6 are identical to the VCG-payment and therefore trivially satisfied by the VCG-Nearest payment.

Is 5-path a single constraint class? First we check if cID 0 covers cID 1, 2 and 4. For this we subtract one of p_l, p_m, p_r from cID 0 and compare the resulting right-hand side (RHS) with the constraint with equal left-hand side (LHS) (after subtraction).

$$
\begin{aligned}
\text{cID 0} &\implies p_m + p_r \geq & b_{m_l} + b_{m_r} - p_l \\
\text{cID 1} &\implies p_m + p_r \geq & \max((b_{m_l} + b_{m_r}), (b_l + b_{m_r})) - b_l \\
& & \text{if } b_l \leq b_{m_l} \text{ cID 0 covers cID 1} \\
& & \text{else if } p_l \leq b_{l_m} \text{ cID 0 covers cID 1} \\
\text{else cID 1 RHS } &= b_{m_r} > & b_{m_l} + b_{m_r} - p_l = \text{cID 0 RHS} - p_l
\end{aligned}
$$

So we see that there are conditions in which cID 1 and by symmetry 4 cannot be reduced by constraint 0. Namely, for cID 1 when $p_l > b_{m_l}$ and when $p_r > b_{m_r}$ for cID 4. Note that since $b_l \geq p_l$ the first condition $b_l > b_{m_l}$ always holds when $p_l > b_{m_l}$.
We will check how cID 0 and cID 2 compare with the same approach as above.

$$
\begin{aligned}
\text{cID 0} \implies & p_l + p_r \geq b_{m_l} + b_{m_r} - p_m \\
\text{cID 2} \implies & p_l + p_r \geq \max((b_{m_l} + b_{m_r}), b_m) - b_m \\
\text{cID 2 is not trivial for } & \max((b_{m_l} + b_{m_r}), b_m) \neq b_m \\
b_{m_l} + b_{m_r} - b_m \leq & b_{m_l} + b_{m_r} - p_m
\end{aligned}
$$

Here we note that cID 2 is always covered by cID 0. This we see as
$0 \leq b_{m_l} + b_{m_r} - p_m$ when the RHS of cID 2 is unequal to zero, and $b_m \geq p_m$.

Next we check if cID 1 and 4 are trivially satisfied by the VCG paymeny. An example confirming that constraint with cID 1 and 4 are not covered by the VCG-payments is $(b_l, b_{m_l}, b_m, b_{m_r}, b_r) = (1, 1, 1, 1, 1)$. Then the VCG-payments are each 0, but the RHS of both 1 and 4 are equal to 1. We therefore reduced the core constraints to the three non-trivial constraints with cID 0, 1 and 4.

cID	payment variables						$WD(cID) - W(b_{cID}, x_{cID})$
0	p_l	$+$	p_m	$+$	p_r	$\geq$	$b_{m_l} + b_{m_r}$
1			p_m	$+$	p_r	$\geq$	b_{m_r}
4	p_l	$+$	p_m			$\geq$	b_{m_l}

Table 7.2: Essential core constraints for 5-path, given $b_{m_l} < p_l$ and $b_{m_r} < p_r$

7.2.2 Blocking core constraints

To further examine 5-path, we look at the blocking constraints. Is it possible to have a total payment greater than $b_{m_l} + b_{m_r}$? If so then there must exist multiple constraints which are blocking simultaneously.

Claim 7.1. cID 0 is always a blocking constraint in 5-path, when l, m, r win the CA.

Proof. If cID 0 is not blocking, then there exists cases, where the sum of payment is greater than the RHS of cID 0: $p_l + p_m + p_r > b_{m_l} + b_{m_r}$. As VCG-nearest payment minimizes the distance, clearly one or more constraints must be blocking. Note that when cID 0 is not blocking, cID 2 cannot be blocking either as it is covered by cID 0. We do a case distinction on other constraints being blocking. We list all other combinations of blocking constraints and show that in no case the sum of payment is greater than the RHS of cID 0.

1. cID 1 and 4 not blocking:
   Then the payment must be equal to the VCG-payment, as those are the remaining constraints on every individual winning bidders payments. This would require that

$$\begin{aligned} p_l^V + p_m^V + p_r^V > b_{m_l} + b_{m_r} \implies \\ \max\big((b_{m_l} + b_{m_r}), (b_m + b_r), (b_{m_l} + b_r)\big) \\ + \max\big((b_l + b_r), (b_l + b_{m_r}), (b_{m_l} + b_{m_r}), (b_{m_l} + b_r)\big) \\ + \max\big((b_{m_l} + b_{m_r}), (b_l + b_m), (b_l + b_{m_r})\big) \\ > b_{m_l} + b_{m_r} + 2(b_l + b_m + b_r) \end{aligned}$$

   And l, m, r winning requires that $\max(b_l + b_{m_r}, b_{m_l} + b_{m_r}, b_{m_l} + b_r) \leq b_l + b_m + b_r$.

   If $(b_{m_l} + b_{m_r})$ were maximal in any of the three cases, we see that the above inequality cannot hold, due to the requirement of l, m, r winning. This observation allows us to simplify the expression removing the case $(b_{m_l} + b_{m_r})$ from every max function.

$$\left.\begin{array}{l} \max\big((b_m + b_r), (b_{m_l} + b_r)\big) \\ + \max\big((b_l + b_r), (b_l + b_{m_r}), (b_{m_l} + b_r)\big) \\ + \max\big((b_l + b_m), (b_l + b_{m_r})\big) \end{array}\right\} > \underbrace{b_{m_l} + b_{m_r} + 2(b_l + b_m + b_r)}_{A}$$

The 12 possible outcomes (from the 3 max functions) are each strictly inferior to A.

(a) $(b_m + b_r) + (b_l + b_r) + (b_l + b_m) = 2(b_l + b_m + b_r)$
(b) $(b_{m_l} + b_r) + (b_l + b_r) + (b_l + b_m) = b_{m_l} + 2b_l + b_m + 2b_r$
(c) $(b_m + b_r) + (b_l + b_{m_r}) + (b_l + b_m) = b_{m_r} + 2b_l + 2b_m + b_r$
(d) $(b_{m_l} + b_r) + (b_l + b_{m_r}) + (b_l + b_m) = b_{m_l} + b_{m_r} + 2b_l + b_m + b_r$
(e) $(b_m + b_r) + (b_{m_l} + b_r) + (b_l + b_m) = b_{m_l} + b_l + 2b_m + 2b_r$
(f) $(b_{m_l} + b_r) + (b_{m_l} + b_r) + (b_l + b_m) = 2b_{m_l} + b_l + b_m + 2b_r$
(g) $(b_m + b_r) + (b_l + b_r) + (b_l + b_{m_r}) = b_{m_r} + 2b_l + b_m + 2b_r$
(h) $(b_{m_l} + b_r) + (b_l + b_r) + (b_l + b_{m_r}) = b_{m_l} + b_{m_r} + 2b_l + 2b_r$
(i) $(b_m + b_r) + (b_l + b_{m_r}) + (b_l + b_{m_r}) = 2b_{m_r} + 2b_l + b_m + b_r$
(j) $(b_{m_l} + b_r) + (b_l + b_{m_r}) + (b_l + b_{m_r}) = b_{m_l} + 2b_{m_r} + 2b_l + b_r$
(k) $(b_m + b_r) + (b_{m_l} + b_r) + (b_l + b_{m_r}) = b_{m_l} + b_{m_r} + b_l + b_m + 2b_r$
(l) $(b_{m_l} + b_r) + (b_{m_l} + b_r) + (b_l + b_{m_r}) = 2b_{m_l} + b_{m_r} + b_l + 2b_r$

Therefore, $p_l^V + p_m^V + p_r^V \leq b_{m_l} + b_{m_r}$ the VCG-payments cannot be the only blocking constraints.

2. cID 1 and 4 blocking:

$$p_l + p_m = b_{m_l} \wedge p_m + p_r = b_{m_r} \implies$$
$$p_l + p_m + p_r = b_{m_l} + b_{m_r} - p_m \leq b_{m_l} + b_{m_r}$$

cID 1 and 4 cannot be simultaneously blocking, as that would result in a payment violating the constraint with cID 0 for $p_m > 0$.

3. cID 1 blocking, 4 not blocking:
Then $p_m + p_r = b_{m_r}$, and in order for cID 0 not to be blocking we require $p_l > b_{m_l}$. The only remaining constraint that could inform p_l is cID 6 wich equals the VCG-payment p_l^V. Hence, we need to check if it is possible that $p_l^V > b_{m_l}$.

$$\begin{aligned} p_l^V &= max((b_{m_l} + b_{m_r}), (b_m + b_r), (b_{m_l} + b_r)) - (b_m + b_r) > b_{m_l} \\ &\implies (b_{m_l} + b_{m_r}) \geq max\big((b_m + b_r), (b_{m_l} + b_r)\big) \\ &\implies b_{m_r} > b_m + b_r \implies b_l + b_{m_r} > b_l + b_m + b_r \end{aligned}$$

This contradicts the assumption that l, m, r are winning. The same argument holds by symmetry for cID 4 blocking and cID 1 not blocking.

Because there exists no combination of other blocking constraints which result in a payment greater than $b_{m_l} + b_{m_r}$, we conclude that cID 0 is always blocking. □

This tells us that the social welfare of the VCG-Nearest payment is always equal to $b_{m_l} + b_{m_r}$ in the SMCA class 5-path.

Claim 7.2. The core constraints with cID 1 and 4 are always covered by cID 0. In other words, the conditions for cID 0 to be covered by one or the other, $b_{m_l} < p_l$ for cID 1 and $b_{m_r} < p_r$ for cID 4, are never satisfied.

Proof. We assume that the constraint 1 is not covered by cID 0. The condition for cID 1 to not being covered by cID 0 is $p_l > b_{m_l}$. We deduce by contradiction that $p_l \leq b_{m_l}$. Having proven claim 7.1 we know that cID 0 is blocking and therefore $p_l + p_m + p_r = b_{m_l} + b_{m_r}$. The constraint cID 1 states that $p_m + p_r \geq b_{m_r}$. After subtracting cID 1 from cID 0 we are left with $p_l \leq b_{m_l}$, contradicting that cID 1 is not covered by cID 0. The same argument concludes for cID 4 that $p_r \geq p_{m_r}$ □

Therefore, the core constraints of the class 5-path can be reduced to the single effective constraint cID 0. By theorem 6.2 there are no over-bidding strategies in the SMCA class 5-path. To summarize, 5-path is an example of a SMCA which allows for three winners, but can be reduced to a single effective core constraint.

7.3 bull-graph

Let us analyze the CA class characterized by the bull graph (drawn in figure 7.1). Is it a single constraint class like 5-path? For this let us start by checking if the core constraints, when there are three winners, can be reduced to a single constraint.

We list the core constraints on the winning bidders. Similarly to 5-path we name the winning bidders l, m, r where the left and right horns of the bull map to l, r and the nose maps to m. The losing bids (eyes) we map to m_l for the left and m_r for the right.

	payment variables						$WD(cID) - W(b_{cID}, x_{cID})$
0	p_l	+	p_m	+	p_r	$\geq$	$\max(b_{m_l}, b_{m_r})$
1	0	+	p_m	+	p_r	$\geq$	$\max\left(b_{m_l}, (b_l + b_{m_r})\right) - b_l$
2	p_l	+	0	+	p_r	$\geq$	$\max(b_{m_l}, b_m, b_{m_r}) - b_m$
3	0	+	0	+	p_r	$\geq$	$p_r^V = \max\left(b_{m_l}, (b_l + b_{m_r}), (b_l + b_m)\right) - (b_l + b_m)$
4	p_l	+	p_m	+	0	$\geq$	$\max\left((b_{m_l} + b_r), b_{m_r}\right) - b_r$
5	0	+	p_m	+	0	$\geq$	$p_m^V = \max\left((b_l + b_r), (b_l + b_{m_r}), (b_{m_l} + b_r)\right) - (b_l + b_r)$
6	p_l	+	0	+	0	$\geq$	$p_l^V = \max\left((b_{m_l} + b_r), (b_m + b_r), b_{m_r}\right) - (b_m + b_r)$

Table 7.3: Core constraints of bull, given the winning assignment (l, m, r)

7.3.1 Reducing core constraints

To check if a core constraint covers another, we compare the lower-bounding RHS of the constraints after adding or subtracting something to get an equal LHS. If for two resulting RHSs one is greater, it covers the other constraint. For the VCG-Nearest payment the constraints on single payments (cIDs 3, 5 and 6) are trivially satisfied, as they are equivalent to the VCG-payment. We begin by comparing the non-trivial constraints (cIDs 0, 1, 2 and 4) against each other. Note that the covers relationship is transitive, i.e. if x covers y and y covers z, then x covers z. This follows from the transitivity of the $\geq$ relation.

Like for 5-path, cID 2 is covered by cID 0, when its RHS is greater zero:

$$\begin{aligned}
\text{cID 0} \implies p_l + p_r &\geq \max(b_{m_l}, b_{m_r}) - p_m \\
\text{cID 2} \implies p_l + p_r &\geq \max(b_{m_l}, b_m, b_{m_r}) - b_m \\
\text{cID 2 is not trivial for } \max(b_{m_l}, b_m, b_{m_r}) &\neq b_m \\
\max(b_{m_l} b_{m_r}) - b_m &\leq \max(b_{m_l}, b_{m_r}) - p_m
\end{aligned}$$

Hence, cID 2 can be removed, as it does not additionaly restrict the core given cID 0. We continue by checking how cID 0 compares to cIDs 1 and 4. Like for 5-path we can again take advantage of the symmetry between l, r and m_l, m_r. We check how cID 0 compares with cID 1.

$$\begin{aligned}
\text{cID 0} \implies p_m + p_r &\geq \max(b_{m_l}, b_{m_r}) - p_l \\
\text{cID 1} \implies p_m + p_r &\geq \max\left(b_{m_l}, (b_l + b_{m_r})\right) - b_l \\
\text{cID 0 covers cID 1 if} \max(b_{m_l}, b_{m_r}) - p_l &\geq \max\left(b_{m_l}, (b_l + b_{m_r})\right) - b_l \\
\implies \text{when } b_{m_r} + p_l &\leq b_{m_l} \text{ cID 0 covers cID 1} \\
\text{And vice versa, if } b_{m_r} + p_l &\geq b_{m_l} \text{ cID 1 covers cID 0}
\end{aligned}$$

By symmetry cID 0 covers cID 4 if $b_{m_l} + p_r \leq b_{m_r}$, and that cID 4 covers cID 0 when $b_{m_l} + p_r \geq b_{m_r}$.

Among the non-trivial constraints, what remains to be done is the comparison of cID 1 and cID 4. We do this by subtracting p_r, respectively p_l s.t. the LHS is equal to p_m for both constraints. In this chapter we write $\gtreqless$ if we don't know if the relation is $\geq$ or $\leq$.

$$\begin{aligned}
\text{cID 1} \quad \implies p_m &\geq \max(b_{m_l}, (b_l + b_{m_r})) - b_l - p_r \\
\text{cID 4} \quad \implies p_m &\geq \max(b_{m_r}, (b_r + b_{m_l})) - b_r - p_l \\
\text{Comparing the RHSs:}& \\
\max(b_{m_l}, (b_l + b_{m_r})) - b_l - p_r &\gtreqless \max(b_{m_r}, (b_r + b_{m_l})) - b_r - p_l \\
\text{Adding } (p_l + p_r) \implies& \\
\max(b_{m_l}, (b_l + b_{m_r})) + p_l - b_l &\gtreqless \max(b_{m_r}, (b_r + b_{m_l})) + p_r - b_r
\end{aligned}$$

We differentiate three cases:

1. $b_{m_l} \geq b_l + b_{m_r}$, then cID 4 covers cID 1
2. $b_{m_r} \geq b_r + b_{m_l}$, then cID 1 covers cID 4
3. $b_{m_l} \leq b_l + b_{m_r} \;\wedge\; b_{m_l} \leq b_l + b_{m_r}$: In this case cID 1 covers cID 4 if $b_{m_r} + p_l \geq b_{m_l} + p_r$, and cID 4 covers cID 1 if $b_{m_r} + p_l \leq b_{m_l} + p_r$.

Thus, we observe that either cID 1 or 4 always covers the other. So far we showed that each of the cIDs 0, 1, 4 may cover the other two, depending on the bids. Next we study what may be the blocking core constraints.

7.3.2 Blocking core constraints

We show that, for SMCA represented by the bull-graph, it is possible to have total payment greater $\max(b_{m_l}, b_{m_r})$. That is to say multiple constraints (including the constraints on single payments) can be simultaneously blocking, as no single constraint has RHS greater $\max(b_{m_l}, b_{m_r})$.

sum of VCG payments We begin by checking if the sum of VCG payments could result in greater payment.

$$p_l^V + p_m^V + p_r^V > \max(b_{m_l}, b_{m_r}) \implies$$

$$\left.\begin{array}{l}\max\big(b_{m_r}, (b_m + b_r), (b_{m_l} + b_r)\big)\\ +\max\big((b_l + b_r), (b_l + b_{m_r}), (b_{m_l} + b_r)\big)\\ +\max\big(b_{m_l}, (b_l + b_m), (b_l + b_{m_r})\big)\end{array}\right\} > \max(b_{m_l}, b_{m_r}) + 2(b_l + b_m + b_r)$$

We can eliminate the cases $b_{m_r} \geq (b_m + b_r)$ and $b_{m_l} \geq (b_l + b_m)$, as they cannot be satisfied when l, m, r are the winners of the auction, which requires $\max(b_l + b_{m_r}, b_{m_l} + b_r) \leq b_l + b_m + b_r$. Hence we simplify to:

$$p_l^V + p_m^V + p_r^V > \max(b_{m_l}, b_{m_r}) \implies$$

$$\left.\begin{array}{l}\max\big((b_m + b_r), (b_{m_l} + b_r)\big)\\ +\max\big((b_l + b_r), (b_l + b_{m_r}), (b_{m_l} + b_r)\big)\\ +\max\big((b_l + b_m), (b_l + b_{m_r})\big)\end{array}\right\} > \underbrace{\max(b_{m_l}, b_{m_r}) + 2(b_l + b_m + b_r)}_{B}$$

Those are the same potential outcomes as listed in the 5-path subsection on blocking constraints 7.2.2. What changed is that we compare to B which is less than $A = B + \min(b_{m_l}, b_{m_r})$. We refrain from listing all cases again. Instead we give one example which is always less than B to show the argumentation and list the three cases which may be greater B. Case (f) (as listed in 7.2.2) is always lesser than B. We assume otherwise and derive a contradiction.

$$\begin{aligned} 2b_{m_l} + b_l + b_m + 2b_r >&\ \max(b_{m_l}, b_{m_r}) + 2(b_l + b_m + b_r) \\ \implies 2b_{m_l} >&\ \max(b_{m_l}, b_{m_r}) + b_l + b_m \\ \implies b_{m_l} >&\ b_l + b_m \\ \implies b_{m_l} + b_r >&\ b_l + b_m + b_r \end{aligned}$$

As we must have $b_l + b_m + b_r \geq b_{m_l} + b_r$ for there to be three winners, this implies that (f) $< B$. Here listed are the three terms which – depending on the bids – may be greater than B:

$$\begin{array}{ll} (h) & (b_{m_l} + b_r) + (b_l + b_r) + (b_l + b_{m_r}) = b_{m_l} + b_{m_r} + 2b_l + 2b_r \\ (j) & (b_{m_l} + b_r) + (b_l + b_{m_r}) + (b_l + b_{m_r}) = b_{m_l} + 2b_{m_r} + 2b_l + b_r \\ (l) & (b_{m_l} + b_r) + (b_{m_l} + b_r) + (b_l + b_{m_r}) = 2b_{m_l} + b_{m_r} + b_l + 2b_r \end{array}$$

We see that $(h), (j)$ and (l) each require that $b_{m_l} \geq b_m \wedge b_{m_r} \geq b_m$. This leaves only the cases derived the from VCG-payment of m: $\max\big((b_l + b_r), (b_l + b_{m_r}), (b_{m_l} + b_m)\big)$. Further we observe that cases (j) and (l) are symmetric, as one could swap b_l with b_r and b_{m_l} with b_{m_r} and get the other case.

Let us look at (h) precisely. The conditions to get (h) are: $b_l \geq b_{m_l} \geq b_m$, $b_r \geq b_{m_r} \geq b_m$. Assuming $(h) > B$ we get:

$$\begin{aligned} b_{m_l} + b_{m_r} + 2b_l + 2b_r > B \\ \implies \min(b_{m_l}, b_{m_r}) > 2b_m \end{aligned}$$

To verify that under the conditions of (h) the sum of payments is greater than $\max(b_{m_l}, b_{m_r})$, we construct an example.
Consider bid profile $b = (b_l, b_{m_l}, b_m, b_{m_r}, b_r) = (6, 5, 1, 5, 6)$ which has VCGN payment outcome $p(b) = (4, 0, 1, 0, 4)$. The payments sum up to $9 > 5 = \max(b_{m_l}, b_{m_r})$ confirming the hypothesis. We wonder if we can reduce the total payment by over-bidding one of the winners in a way that reduces the over-bidders payment. The only player which can reduce the sum of payment by over-bidding is m. This is the case, as by the conditions of (h), p_m is the only variable controlled by one of the winning bidders which determines if the sum of payments is greater than B or not ($\min(b_{m_l}, b_{m_r}) > 2b_m$). As predicted, we observe no change in the outcome after over-bidding of b_l or b_r. Over-bidding b_m on the other hand does indeed reduce the sum of payment up to $\max(b_{m_l}, b_{m_r})$. However, in the examples we studied, we observe an increasing payment $p_m = \min\big(b_m, \max(b_{m_l}, b_{m_r})\big)$. This would indicate that there is no incentive to overbid for player m.

Let us do the same analysis in case (j), we will deduce symmetric results for case (l).

Conditions of (j) are:

$\max(b_l + b_{m_r}, b_{m_l} + b_r) \leq b_l + b_m + b_r$	l, m, r winning
$(b_{m_l} \geq b_m) \wedge (b_{m_r} \geq b_m)$	common conditions of $(h), (j), (l)$
$(b_{m_r} \geq b_r) \wedge (b_{m_r} - b_r \geq b_{m_l} - b_l)$	specific conditions to (j)

Assuming $(j) > B$ we get:

$$
\begin{aligned}
& b_{m_l} + 2b_{m_r} + 2b_l + b_r > B \\
\implies & b_{m_l} + 2b_{m_r} - \max(b_{m_l}, b_{m_r}) > 2b_m + b_r \\
\implies & \begin{cases} 2b_{m_r} > 2b_m + b_r & \text{if } b_{m_l} \geq b_{m_r} \\ b_{m_l} + b_{m_r} > 2b_m + b_r & \text{if } b_{m_l} \leq b_{m_r} \end{cases}
\end{aligned}
$$

Again we confirm that this results in a sum of payments greater than $\max(b_{m_l}, b_{m_r})$ by example. If $b_{m_l} \geq b_{m_r}$, for bids $(b_l, b_{m_l}, b_m, b_{m_r}.b_r) = (6, 5, 3, 6, 4)$ the VCGN payment equals $(2, 0, 3, 0, 3)$. Then the sum of payments 8 is greater than $\max(b_{m_l}, b_{m_r}) = 6$. For the second case $b_{m_l} \leq b_{m_r}$, an example is $p(6, 7, 4, 5, 3) = (3, 0, 4, 0, 1)$. Here the payments sum to 8 while the greatest losing bid amounts to 7.

Like before, we are interested to observe how over-bidding affects the sum of payment. In the examples we explored, over-bidding b_m behaved according to case (f). Over-bidding reduced the sum of payment but increased p_m to the detriment of the over-bidder. As the conditions on $(j) > B$ are characterized by $2b_m + b_r$, we expected that over-bidding b_r would equally lead to a decrease in total payment. Against our expectations, over-bidding b_r for (j) (or b_l in case (l)) to go from $(j) >$ B to $(j) <$ B did not reduce the sum of payment. In fact it had no effect on the outcome. One explanation would be that in this case, there is a dominating blocking constraint of cID 1 and 6 or cID 3 and 4. This dominating constraint would cover the sum of VCG payments. We will verify this when comparing cID 1 and 6 blocking to the sum of VCG payments.

cID 1 or 4 blocking By symmetry we only look at the case when cID 1 is blocking. This constraint limits p_m, p_r. The constraint cID 6 lower-bounding only p_l could be blocking together with cID 1. When cID 1 and 6 are blocking, we begin by comparing the sum of the RHSs to the RHS of cID 0 $\max(b_{m_l}, b_{m_r})$. We assume that cID 1 covers 0, which holds when $b_{m_r} + p_l \geq b_{m_l}$.

$$
\begin{aligned}
& p_l^V + \max(b_{m_l}, b_l + b_{m_r}) - b_l = p_l^V + b_{m_r} \gtreqless \max(b_{m_l}, b_{m_r}) \\
\implies & \max\big((b_{m_l} + b_r), (b_m + b_r), b_{m_r}\big) + b_{m_r} \gtreqless b_m + b_r + \max(b_{m_l}, b_{m_r})
\end{aligned}
$$

We distinguish cases based on the maximal value of the LHS:

- $(b_{m_l} + b_r) = \max\big((b_{m_l} + b_r), (b_m + b_r), b_{m_r}\big)$:

$$\implies b_{m_l} \geq b_m \wedge b_{m_l} + b_r \geq b_{m_r}, \text{ then}$$
$$b_{m_l} + b_r + b_{m_r} > b_m + b_r + \max(b_{m_l}, b_{m_r}) \iff \min(b_{m_l}, b_{m_r}) > b_m$$

- $(b_m + b_r) = \max\big((b_{m_l} + b_r), (b_m + b_r), b_{m_r}\big)$:

$$\implies b_{m_l} \leq b_m \wedge b_{m_l} + b_r \geq b_{m_r}, \text{ then}$$
$$b_m + b_r + b_{m_r} \leq b_m + b_r + \max(b_{m_l}, b_{m_r}), \text{ as } b_{m_r} \leq \max(b_{m_l}, b_{m_r})$$

- $b_{m_r} = \max\big((b_{m_l} + b_r), (b_m + b_r), b_{m_r}\big)$:

$$\implies b_{m_l} \leq b_m \wedge b_{m_l} + b_r \leq b_{m_r}, \text{ then}$$
$$2b_{m_r} \leq b_m + b_r + \max(b_{m_l}, b_{m_r}), \text{ as } b_{m_r} \leq b_m + b_r \text{ due to } l, m, r \text{ winning}$$

The only case when cID 1 covers cID 0 ($b_{m_r} + p_l \geq b_{m_l}$) where blocking cIDs 1 and 6 result in total payment greater $\max(b_{m_l}, b_{m_r})$ is when

$$b_{m_l} \geq b_m \wedge b_{m_l} + b_r \geq b_{m_r} \wedge \min(b_{m_l}, b_{m_r}) > b_m$$

Next we check if blocking cID 1 and 6 may also cover the sum of VCG payments in cases $(f), (l), (j)$. In fact, we find that if cID 0 is covered by cID 1 or 4, then the resulting sum of payment is always greater than the sum of VCG payments. Here we note the case for cID 0 being covered by cID 1 and verify that $b_{m_r} + p_l^V \geq p_l^V + p_m^V + p_r^V$.

$$b_{m_r} > p_m^V + p_r^V \implies$$
$$\underbrace{b_{m_r} + 2b_l + b_m + b_r}_{C} > \begin{cases} \max((b_l + b_r), (b_l + b_{m_r}), (b_{m_l} + b_r)) \\ +\max(b_{m_l}, (b_l + b_m), (b_l + b_{m_r})) \end{cases}$$

One can list all 9 possible outcomes of the two max functions and verify that it is always strictly smaller than C. To do so we make use of the assumption that l, m, r are the winning bidders: $b_l + b_m + b_r \geq b_{m_l} + b_r \implies b_l + b_m \geq b_{m_l}$ and similarly for $b_{m_r} \leq b_m + b_r$. Additionally, we make use of the assumption that cID 0 is covered by cID 1 $b_{m_r} + p_l \geq b_{m_l} \implies b_{m_r} + b_l \geq b_{m_l}$. If we drop this last assumption for some of the cases the sum of VCG payments is greater than the RHS of cID 1 + the RHS of cID 6.

Let us write down the cases where the sum of VCG payment is not covered by blocking cIDs 1 and 6 and check if those match with cases $(f), (j)$ or (l). We will see that the cases when the sum of VCG payment is not covered by blocking cIDs 1 and 6, are different from $(f), (j)$ or (l). Therefore, we know that when the sum of VCG payment is not covered by cIDs 1 and 6, then it is covered by cID 0. Or differently put, whenever the VCG payments are not covered by cID 0, they are covered by cIDs 1 and 6 or 3 and 4.

The two cases which require the assumption that cID 0 is covered by cID 1 to be less than C, ask for $b_{m_l} \geq b_l \wedge (b_{m_l} + b_r \geq b_l + b_{m_r})$. The first case has the stronger conditions $b_{m_l} \geq b_l + b_m \wedge b_{m_l} \geq b_l + b_{m_r}$. Then we get

$$\begin{aligned} C \gtreqless 2b_{m_l} + b_r &\implies b_{m_r} + 2b_l + b_m \gtreqless 2b_{m_l} \\ &\text{we know that } b_l + b_m \geq b_{m_l} \\ \text{if } b_{m_l} - (b_{m_r} + b_l) > (b_l + b_m) - b_{m_l} &\implies C < 2b_{m_l} + b_r \\ &\text{else } C \geq 2b_{m_l} + b_r \end{aligned}$$

This states that under those conditions ($b_{m_l} \geq b_l + b_m \wedge b_{m_l} \geq b_l + b_{m_r} \wedge 2b_{m_l} > 2b_l + b_{m_r} + b_m$) the sum of VCG payments is greater than the RHSs of cID 1 + cID 6. Let us check if it is then covered by RHS of (cID 3 + cID 4). To do so we compare

$$\begin{aligned} b_{m_l} + p_r^V \gtreqless p_l^V + p_m^V + p_r^V &\iff b_{m_l} \gtreqless p_l^V + p_m^V \\ \implies \underbrace{b_{m_r} + b_l + b_m + 2b_r}_{D} \gtreqless &\begin{cases} \max((b_{m_l} + b_r), (b_m + b_r), b_{m_r}) \\ + \max((b_l + b_r), (b_l + b_{m_r}), (b_{m_l} + b_r)) \end{cases} \end{aligned}$$

We apply the constraints from above:
$b_{m_l} \geq b_l + b_m \wedge b_{m_l} \geq b_l + b_{m_r}$, and $2b_{m_l} > 2b_l + b_m + b_{m_r}$. Then the sole possibility of the max functions results in

$$\begin{aligned} D < 2b_{m_l} + 2b_r &\implies b_{m_r} + b_l + b_m < 2b_{m_l} \\ &\text{this follows from } 2b_{m_l} > 2b_l + b_m + b_{m_r} \end{aligned}$$

It is not covered by blocking cIDs 3 and 4. But comparing with the conditions from $(f), (j)$ and (l) we see that it must be covered by cID 0. The cases $(f), (j)$, or (l) require $b_{m_r} \geq b_m$ which results in a contradiction to l, m, r winning the auction as follows

$$2b_{m_l} > 2b_l + b_m + b_{m_r} \geq 2b_l + 2b_m \implies b_{m_l} + b_r > b_l + b_m + b_r$$

Hence this represents a case where the sum of the three VCG payments is covered by cID 0, but not by 1 or 4.

We perform the same analysis on the second case not covered by blocking cIDs 1 and 6. This case has the conditions $b_{m_l} \geq b_l \wedge (b_{m_l} + b_r \geq b_l + b_{m_r})$ and $b_l + b_m \geq (b_{m_l}) \wedge b_m \geq b_{m_r}$.

$$C \gtreqless b_l + b_{m_l} + b_m + b_r \implies b_{m_r} + b_l \gtreqless b_{m_l}$$

In order to have have a RHS greater C we need to assume $b_{m_r} + b_l < b_{m_l}$

Applying those constraints to check if those are covered by cID 3 and 4 we end up with

$$\begin{aligned} D &\gtreqless \max((b_{m_l}, b_m) + 2b_r + b_{m_l} \\ \text{Considering the cases which may be in (f), (j), (l) requires } b_{m_l} &\geq b_m \\ \implies D &\gtreqless 2b_r + 2b_{m_l} \quad \text{same as above.} \end{aligned}$$

Hence we showed that in the SMCA class represented by the bull-graph, the cIDs 3, 5, 6 never form blocking constraint together, as the sum of VCG-payments is always covered by either cID 0, cIDs 1 and 6, or cIDs 3 and 4.

Finally, we exclude the possibility of both cID 1 and 4 being blocking at the same time. Recall when we checked the conditions where cID 1 covers 4 and vice versa. We have seen that one of the two constraints always covers the other. Therefore, for specific bids it cannot be that we require both constraints 1 and 4 to express the core.

7.4 Outlook

We showed that the bull-graph represents a class of SMCA with three winners which is not a single core constraint class. This makes it an interesting class to examine with regards to over-bidding. Our efforts so far could neither confirm nor disprove the existence of over-bidding strategies in the bull class. We characterized the conditions of the non-trivial core constraints with cIDs (0,1,4) to cover one another. Further, we confirmed that there exist outcomes with sum of payment greater than $\max(b_{m_l}, b_{m_r})$. This indicates that multiple core constraints can be blocking simultaneously (one of 0, 1 and 4 combined with some of the constraints on single payments 3,5 and 6). With the question of over-bidding in mind, we know that, for bids in ranges where a single constraint covers the others, there is no over-bidding. It would be interesting to study if one can find over-bidding strategies where the over-bid leads to a change in which constraint(s) are blocking. This can never happen in single core constraint classes.

From the seven LLLG+1 classes (see figure 7.1) four ({0,1},{1,1},{0,2},{1,2}) could be interesting with regards to over-bidding and have not been examined. They represent open cases of SMCA classes on five bidders with three winners, which potentially include examples of over-bidding under VCG-Nearest payment. Analyzing those would further our understanding on when over-bidding can or can not occur. Studying those along with the bull class appears like the logical next step on the quest to find a minimal example of over-bidding.

Chapter 8

Conclusion

This Master's Thesis sets the foundations for reasoning about over-bidding in single-minded combinatorial auctions (CA). We began by defining over-bidding strategies and setup the search for a minimal example of over-bidding in single-minded CA under the VCG-Nearest payment. Along the way we gained valuable insight concerning existence criteria of over-bidding strategies.

8.1 Results

Our model focused on single-minded combinatorial auctions (SMCA) mechanisms $\mathcal{M} = (X, P)$, with allocation algorithm X maximizing the reported social welfare and core selecting payments P satisfying voluntary participation.

- In section 4.4 we extend the proof by Bosshard18 [4] that proxy and proportional payments are non-decreasing, by taking into account how over-bidding may effect the core.
- We show that over-bidding a losing bid, such that the over-bid value turns it into a winning bid, is never profitable (theorem 4.2).
- For non-decreasing payment P, we show that there exist no over-bidding strategies in SMCA (corollary 4.4). This implies that for non-decreasing payment P, there exist no Nash equilibria containing over-bidding in SMCA.
- We introduced SMCA classes, grouping instances with same behaviour (chapter 5). We achieved this by mapping CA to graphs where a bid is represented by a vertex and two bids are connected by an edge, if the respective bundles the bids where placed on intersect in some items. Hence, we can list the finite CA classes on n bidders by their finite representations of non-isomorphic graphs on n vertices.
- Chapter 6 describes our result on single core constraint classes. If the core constraints can be reduced to a single effective constraint, the VCGN

payment is non-decreasing (theorem 6.2). The corollary 6.7 states that CA classes, which have no possible allocation with more than two winners, are single core constraint classes.

- Based on the result from corollary 6.7, and the graph representation of CA classes, we know that CAs with four or less bidders are robust to over-bidding under the VCGN payment. In chapter 7 we list the nine CA classes on five bidders, which allow for three or more winners. For four of them, we argue that they are robust against over-bidding. The remaining five classes are the minimal candidates, for which we could not eliminate the existence of over-bidding in the scope of this thesis.

8.2 Future work

To continue the search for the minimal example of over-bidding in SMCA under VCGN payment, picking up where chapter 7 ended appears promising. Another way to find smaller examples of over-bidding, could be to try to simplify the over-bidding example on eleven bidders (table 2.2). Maybe not all bidders are necessary to provoke over-bidding. If this fails, one could directly study the eleven bidders example. Maybe one can reduce the number of core constraints to just a few critical ones? Or, observe how the core is affected by the profitable over-bid?

When the occurrence of over-bidding is better understood and more examples are known, new directions of questioning become interesting. We end this thesis with a list of potential research questions.

- Is over-bidding a viable strategy, when the players have incomplete information?
- Do over-bidding strategies occur in Nash equilibria?
- Can a single bidder reliably abuse over-bidding to their benefit?
- Is over-bidding a significant issue that needs to address by mechanism design, or does it occur to infrequently and with too little impact?
- What changes when we consider multi-minded CA?
- Can one leverage computational approaches to tackle over-bidding?

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